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Please answer all questions, using without proof/calculation everything that was covered in class.

PROBLEM 1

THE ECONOMY

- two goods, A, X
- One consumer
- Two firms,1 and 2

Firm 1 produces good A out of good X with production function

$$A_1 = 2\sqrt{X_1}, X_1 \geq 0 \qquad (1)$$

where X_1 is the quantity of good X used as input in the production of good A, and A_1 is the quantity produced of good A.

Firm 2 produces good A out of good X with technology described by the production function

$$A_2 = \frac{2X_2}{1+X_1}, X_2 \geq 0 \qquad (2)$$

where X_2 is the quantity of good X used as input in the production of good A, and A_2 is the quantity produced of good A.

Consumer

- Consumption set R^2_+ $_{A,X}$
- Endowment vector $\omega = [0, \bar{X}], \bar{X} > 0$
- Ownership shares: $\theta_1 = 1 = \theta_2$
- Utility function $u(A, X, A_1, X_1, A_2, X_2) = AX$

where X is the quantity consumed of good X, and A is the quantity consumed of good A.

$$
\boxed{\text{The pareto problem of this economy}}
$$

objective function

$u = AX$

Feasible set

$$S = \left\{ z \in \mathbb{R}_+^6 : X + X_1 + X_2 \leq \bar{X}, A_1 + A_2 \geq A, A_1 = 2\sqrt{X_1}, A_2 = \frac{2X_2}{1+X_1} \right\} \tag{3}$$

for $z = \left(A, X, A_1, X_1, A_2, X_2\right)$

QUESTIONS

(The answers might depend on the values of the parameters)

1.Compute all competitive equilibria. If there is more than one, are they pareto-ranked?

2.Is any of the equilibria pareto efficient, i.e., are they global maxima of the pareto problem (3)?

3.Suppose that the two firms merge. Compute all competitive equilibria. Are they efficient?

4.Suppose instead that firm 2 is given property rights to a pollution-free environment, and that there is a market for emission rights. Compute all competitive equilibria. Are they efficient? (We refer to the negative externality as pollution; the rights to pollute are called emissions rights).

5. Suppose instead that price controls of the form $\frac{\text{price of } X}{\text{price of A}} \geq \psi$ are introduced, where $\psi > 0$ is some constant. For which values of ψ do equilibria exist? For which values of ψ do equilibria exist and are all efficient?

6. Suppose instead that a tax $t > 0$ per unit of input used is imposed on firm 1, and the tax revenue is given to the consumer with a lump-sum transfer. Compute all competitive equilibria. Compute the values of t ,if any, for which all equilibria are efficient.

ANSWERS TO PROBLEM 1

1.COMPETITIVE EQUILIBRIUM WITH TWO FIRMS

1.NAME THE PRICE OF EACH GOOD

$p =$ price of commodity A,w=price of commodity X.Normalize $p = 1$

2. DEFINE CONSUMER INCOME

$$M = w\bar{X} + \Pi_1 + \Pi_2 \tag{4}$$

3. SOLVE THE OPTIMIZATION PROBLEM OF THE CONSUMER

max $u = \log(A) + \log(X)$,subject to $A + wX \leq M, A \geq 0, X \geq 0$
Variables: A, X
parameters: w, M
Conditions on parameters: $w > 0, M > 0$

The solutions are

$$\left(A, X\right) = \left(\frac{M}{2}, \frac{M}{2w}\right) \tag{5}$$

4.1. SOLVE THE OPTIMIZATION PROBLEM OF FIRM 1

max $\Pi_1 = A_1 - wX_1$
subject to $X_1 \geq 0, A_1 \geq 0, A_1 = 2\sqrt{X_1}$
Variables:A_1, X_1
parameters: w
conditions on parameters:$w > 0$

The solution is

$$\left(A_1, X_1, \Pi_1\right) = \left(\frac{2}{w}, \frac{1}{w^2}, \frac{1}{w}\right) \tag{6}$$

4.2. SOLVE THE OPTIMIZATION PROBLEM OF FIRM 2

$\max \Pi_2 = A_2 - wX_2$

subject to $A_2 \geq 0, X_2 \geq 0, A_2 = \dfrac{2X_2}{1+X_1}$

Variables: A_2, X_2

parameters: w, X_1

conditions on parameters: $w > 0, X_1 \geq 0$

The solution is

$$\left(A_2, X_2, \Pi_2\right) = \begin{cases} \text{NONE} & \text{if} \quad w < \dfrac{2}{1+X_1} \\ \left(\dfrac{2X_2}{1+X_1}, X_2, 0\right) & \text{if} \quad w = \dfrac{2}{1+X_1} \\ \left(0,0,0\right) & \text{if} \quad w > \dfrac{2}{1+X_1} \end{cases} \tag{7}$$

By (7) and (6) we obtain

$$\left(A_2, X_2, \Pi_2\right) = \begin{cases} \left(0,0,0\right) & \text{if} \quad w \neq 1 \\ \left(X_2, X_2, 0\right) & \text{if} \quad w = 1 \end{cases} \tag{8}$$

By (8) and (6) we obtain the aggregate supply correspondence for good A

$$A_1 + A_2 = \begin{cases} 2/w & \text{if} \quad w \neq 1 \\ 2 + A_2 & \text{if} \quad w = 1 \end{cases} \tag{9}$$

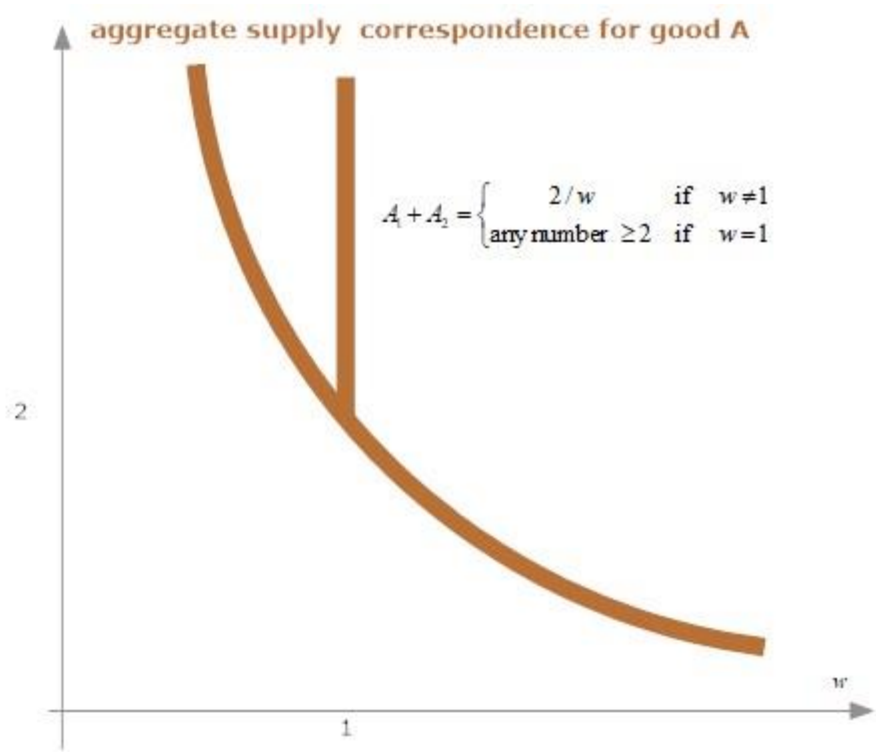


Figure 1 aggregate supply correspondence for good A

By (8),(5),(4),(6) we obtain

$$A = \frac{w\bar{X} + (1/w)}{2} \tag{10}$$

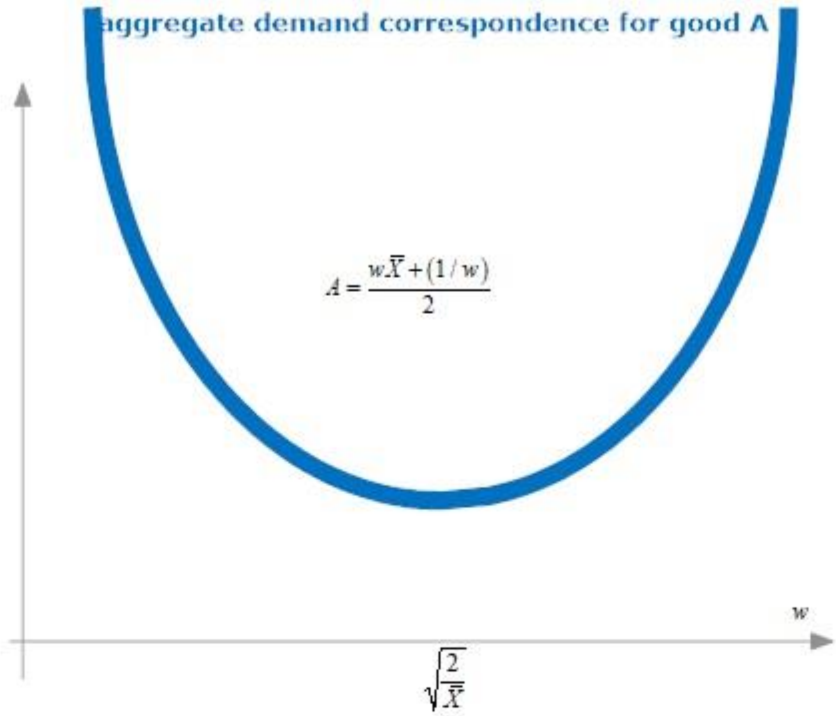


Figure 2 aggregate demand correspondence for good A

5. SOLVE THE EQUILIBRIUM CONDITIONS

$$A = A_1 + A_2 \tag{11}$$

$$X + X_1 + X_2 = \bar{X} \tag{12}$$

There are two types of solutions of the equilibrium equations (11),(12)

$$\boxed{\begin{array}{l} \text{type 1 competitive equilibrium} \\ w = \sqrt{\frac{3}{\bar{X}}} \\ \left(A_2, X_2, \Pi_2\right) = (0,0,0) \\ \left(A_1, X_1, \Pi_1\right) = \left(2\sqrt{\frac{\bar{X}}{3}}, \frac{\bar{X}}{3}, \sqrt{\frac{\bar{X}}{3}}\right) \\ \left(A, X\right) = \left(2\sqrt{\frac{\bar{X}}{3}}, \frac{2\bar{X}}{3}\right) \\ U_{EQ1} = \frac{4}{3\sqrt{3}}\left(\bar{X}\right)^{3/2} \end{array}} \tag{13}$$

$$\boxed{\begin{array}{l} \text{type 2 competitive equilibrium,it exists when } \boxed{\bar{X} \geq 3} \\ w = 1 \\ \left(A_2, X_2, \Pi_2\right) = \left(\frac{\bar{X}-3}{2}, \frac{\bar{X}-3}{2}, 0\right) \\ \left(A_1, X_1, \Pi_1\right) = (2,1,1) \\ \left(A, X\right) = \left(\frac{\bar{X}+1}{2}, \frac{\bar{X}+1}{2}\right) \\ U_{EQ2} = \frac{\left(\bar{X}+1\right)^2}{4} \end{array}} \quad (14)$$

Hence there exist two equilibria when $\bar{X} > 3$.There is a unique equilibrium, given by (13),when $0 < \bar{X} \leq 3$.

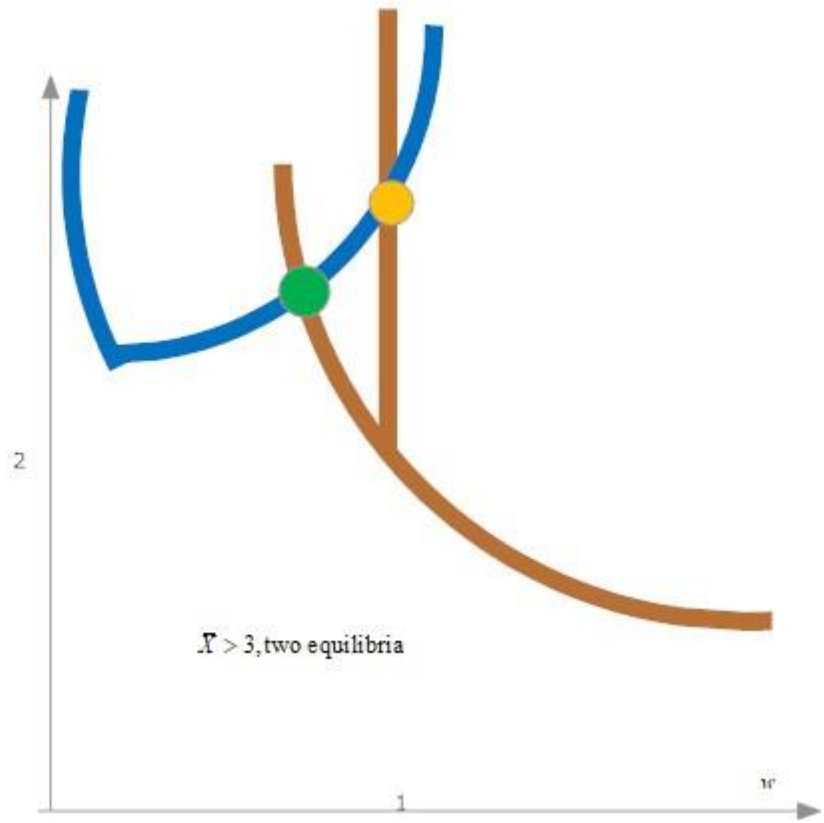


Figure 3 two Equilibria

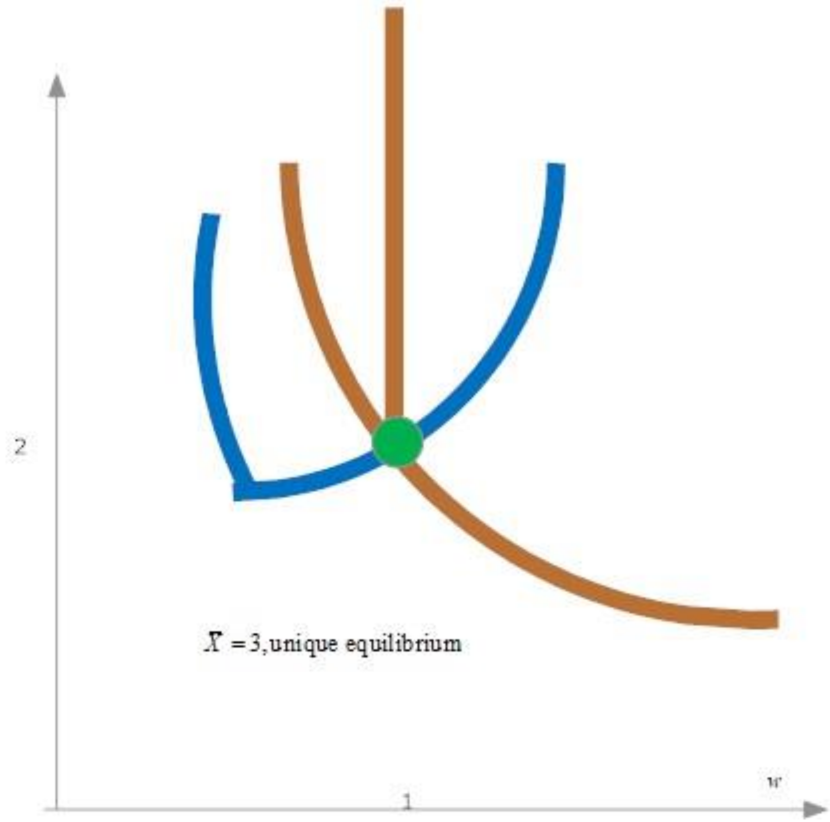


Figure 4 unique equilibrium

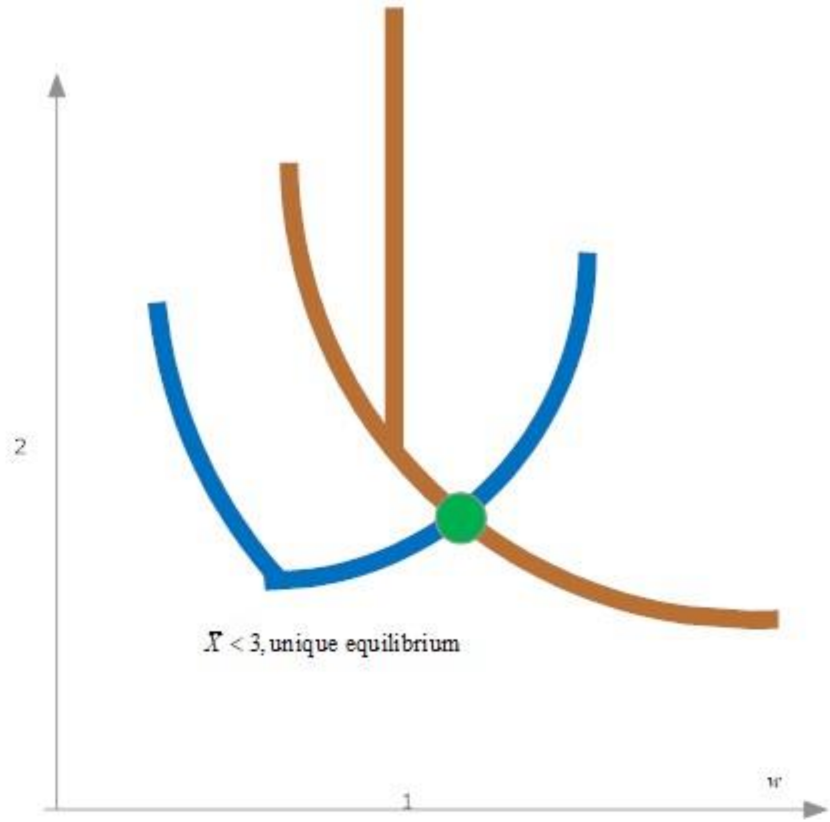


Figure 5 unique equilibrium

In the case $\bar{X} > 3$, the second type equilibrium is better than the first, in the sense that

$$\boxed{U_{EQ2} = \frac{\left(\bar{X}+1\right)^2}{4} > \frac{4}{3\sqrt{3}}\left(\bar{X}\right)^{3/2} = U_{EQ1}, \text{when } \bar{X} > 3} \tag{15}$$

In other words, EQ1 is pareto inefficient, and EQ2 is pareto superior to EQ1.

2.PARETO EFFICIENCY OF EQUILIBRIA WITH TWO FIRMS

The pareto problem (3) is equivalent to

$$\boxed{\begin{array}{l}\text{simplified pareto problem}\\ \max u = AX\\ \text{subject to}\\ \bar{X} - X - X_1 - X_2 \geq 0\\ 2\sqrt{X_1} + \dfrac{2X_2}{1+X_1} - A \geq 0\\ (A, X, X_1, X_2) \in \mathbb{R}_+^4\end{array}} \tag{16}$$

The feasible set is not, in general, convex. For example, when $\bar{X} > 6$, the points $P = \overbrace{[8,0,0,4]}^{A,X,X_1,X_2}, Q = \overbrace{[2\sqrt{2} + \frac{8}{3}, 0, 2, 4]}^{A,X,X_1,X_2}$ are both feasible, but their convex combination $\frac{P+Q}{2} = \overbrace{[\frac{16}{3} + \sqrt{2}, 0, 1, 4]}^{A,X,X_1,X_2}$ is not.

When $\bar{X} > 3$, EQ1 is inefficient because it is pareto-dominated by EQ2; and EQ2 is also inefficient, because it does not satisfy the Fritz John necessary conditions of (16) with $X_1 > 0, X_2 > 0$, namely

$$-\lambda_1 + A = 0, -\lambda_1 + \frac{2\lambda_2}{1+X_1} = 0, -\lambda_1 + \lambda_2\left(\frac{1}{\sqrt{X_1}} - \frac{2X_2}{(1+X_1)^2}\right) = 0, -\lambda_2 + X = 0 \tag{17}$$

When $\frac{64}{27} < \bar{X} \leq 3$, the unique equilibrium EQ1, as given by (13), is inefficient. **EQ1** is the unique solution of the Fritz John necessary conditions of (16) with $X_2 = 0$, namely

$$\begin{gathered}-\lambda_1 + A = 0, -\lambda_1 + \lambda_2\left(\frac{1}{\sqrt{X_1}}\right) = 0, -\lambda_2 + X = 0, -\lambda_1 + \frac{2\lambda_2}{1+X_1} \leq 0\\ \bar{X} = X + X_1, 2\sqrt{X_1} = A, X_2 = 0\end{gathered} \tag{18}$$

EQ1 is, therefore, a candidate efficient point, but it is not efficient because it is dominated by the solution of (16) with the extra constraint $X_1 = 0$, namely the point

$$P = [A = \bar{X}, X = \frac{\bar{X}}{2}, X_1 = 0, X_2 = \frac{\bar{X}}{2}], U_P = \frac{\bar{X}^2}{2} \tag{19}$$

When $\bar{X} \le 1.49$, the unique equilibrium EQ1,as given by (13),is efficient

because EQ1 dominates point P; and because **EQ1** is the only solution of the necessary conditions, since (17) is unsatisfiable when $\bar{X} \le 1.49$ .To see that (17) is unsatisfiable, note that it is equivalent to the conditions

$$\begin{aligned}
&A = \frac{3t^2 - 2t + 1}{t}, X = \frac{(t^2+1)(3t^2-2t+1)}{2t}, X_2 = \frac{(t^2+1)(t-1)^2}{2t} \\
&\bar{X} = \frac{2t^4 - t^3 + 3t^2 - 2t + 1}{t} \\
&\lambda_1 = \frac{3t^2 - 2t + 1}{t}, \lambda_2 = \frac{(t^2+1)(3t^2-2t+1)}{2t}, t = \sqrt{X_1}
\end{aligned} \tag{20}$$

The minimum value of the expression $\frac{2t^4 - t^3 + 3t^2 - 2t + 1}{t}$ is 1.493.Hence for $\bar{X} \le 1.49$ there is no solution of (20),and hence of (17).

3. COMPETITIVE EQUILIBRIUM, MERGER

1.NAME THE PRICE OF EACH GOOD

$p =$ price of commodity A,w=price of commodity X.Normalize $p = 1$

2. DEFINE CONSUMER INCOME

$$M = w\bar{X} + \Pi \tag{21}$$

3. SOLVE THE OPTIMIZATION PROBLEM OF THE CONSUMER

max $u = \log(A) + \log(X)$,subject to $A + wX \le M, A \ge 0, X \ge 0$
Variables: A, X
parameters: w, M
Conditions on parameters: $w > 0, M > 0$

The solutions are

$$(A, X) = \left(\frac{M}{2}, \frac{M}{2w}\right) \tag{22}$$

4.1. SOLVE THE OPTIMIZATION PROBLEM OF THE FIRM

$\max \Pi = A_1 - wX_1 + A_2 - wX_2$

subject to $X_1 \geq 0, A_1 \geq 0, A_1 = 2\sqrt{X_1}, A_2 \geq 0, X_2 \geq 0, A_2 = \dfrac{2X_2}{1+X_1}$

Variables: A_1, X_1, A_2, X_2

parameters: w

conditions on parameters: $w > 0$

The solution is

$$\left(A_1, X_1, A_2, X_2, \Pi\right) = \begin{cases} \text{NONE} & \text{if} \quad w < 2 \\ \left(\dfrac{2}{w}, \dfrac{1}{w^2}, 0, 0, \dfrac{1}{w}\right) & \text{if} \quad w \geq 2 \end{cases} \tag{23}$$

By (23) we obtain the aggregate supply correspondence for good A

$$A_1 + A_2 = \begin{cases} \text{NONE} & \text{if} \quad w < 2 \\ \dfrac{2}{w} & \text{if} \quad w \geq 2 \end{cases} \tag{24}$$

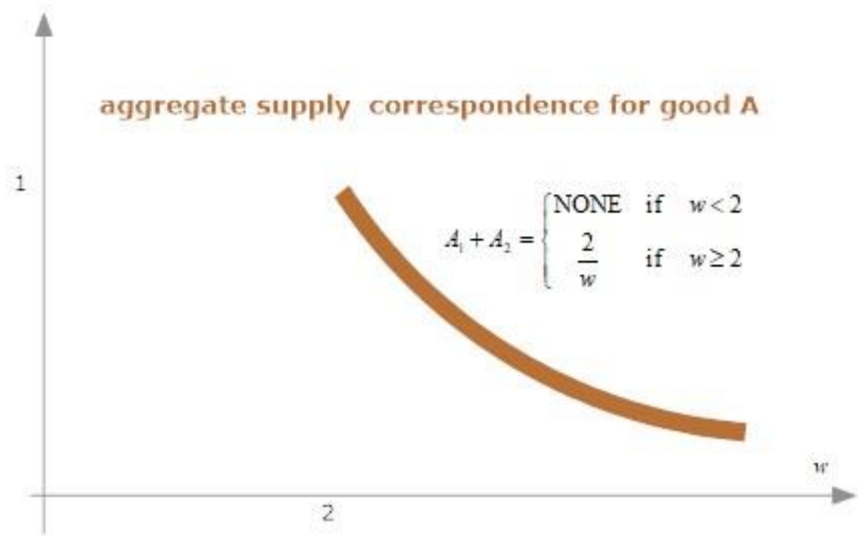


Figure 6 aggregate supply correspondence for good A

5. SOLVE THE EQUILIBRIUM CONDITIONS

$$A = A_1 + A_2 \tag{25}$$

$$X + X_1 + X_2 = \bar{X} \tag{26}$$

$$\boxed{\begin{array}{l} \text{competitive equilibrium,merger} \\ \text{if } \bar{X} > 3/4, \text{no equilibrium exists} \\ \text{if } \bar{X} \le 3/4, \text{then} \\ w = \sqrt{\dfrac{3}{\bar{X}}} \\ \left(A_2, X_2\right) = (0,0) \\ \left(A_1, X_1, \Pi\right) = (2\sqrt{\dfrac{\bar{X}}{3}}, \dfrac{\bar{X}}{3}, \sqrt{\dfrac{\bar{X}}{3}}) \\ \left(A, X\right) = \left(2\sqrt{\dfrac{\bar{X}}{3}}, \dfrac{2\bar{X}}{3}\right) \\ U_{\text{MERGER}} = \dfrac{4}{3\sqrt{3}}\left(\bar{X}\right)^{3/2} \end{array}} \tag{27}$$

Hence merger will lead to the unique, efficient equilibrium, provided that $\bar{X} \le 3/4$ .Nonexistence in the case $\bar{X} > 3/4$ is due to the nonconvexity of the aggregate production set (even though both individual production sets are convex).

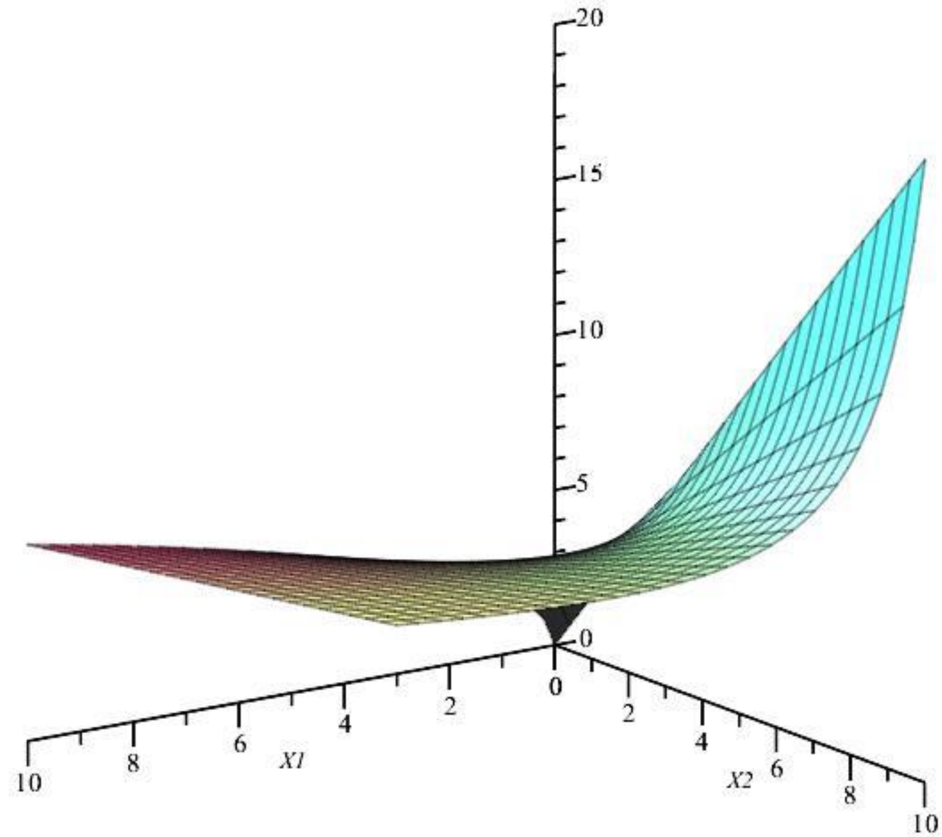


Figure 7 Aggregate production function

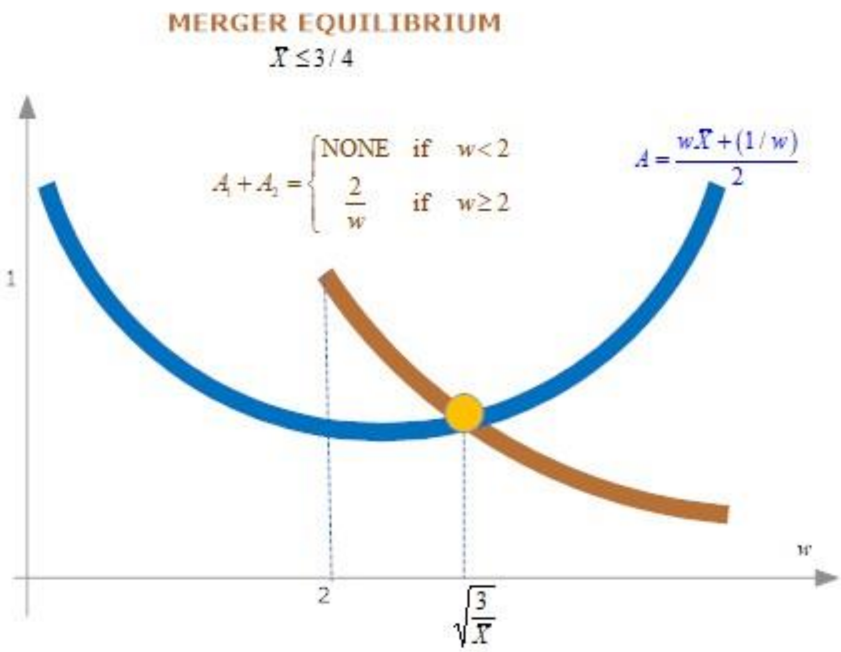


Figure 8 Merger equilibrium

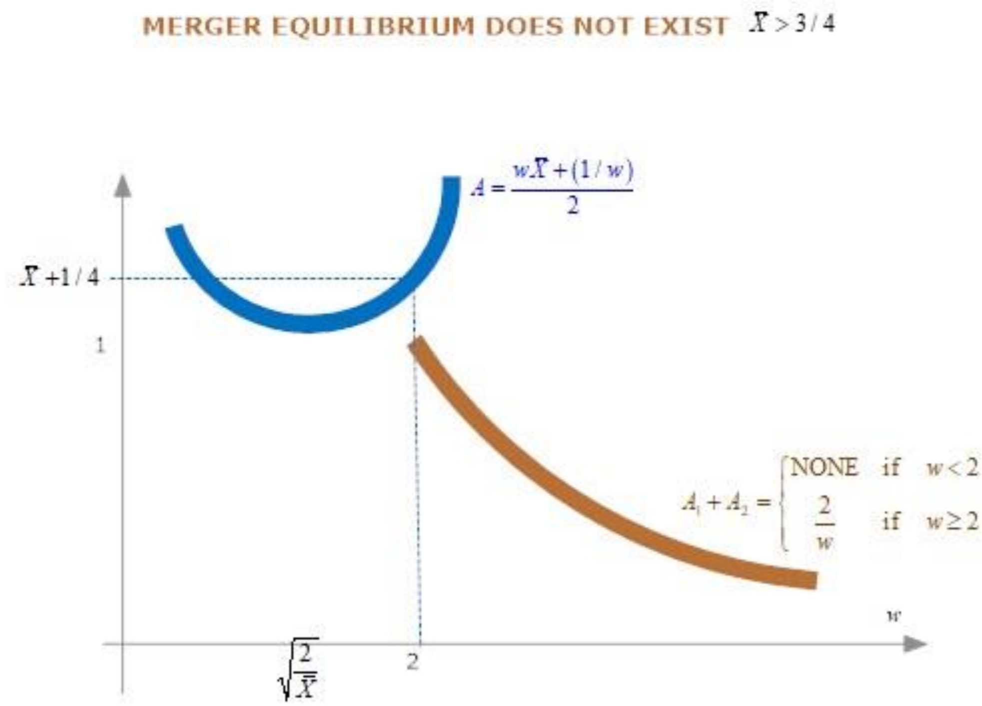


Figure 9 nonexistence of merger equilibrium

4.COMPETITIVE EQUILIBRIUM, TRADABLE EMISSION RIGHTS

1.NAME THE PRICE OF EACH GOOD

p = price of commodity A,w=price of commodity X
q=price of commodity E(emission rights).Normalize $p = 1$

2. DEFINE CONSUMER INCOME

$$M = w\bar{X} + \Pi_1 + \Pi_2 \tag{28}$$

3. SOLVE THE OPTIMIZATION PROBLEM OF THE CONSUMER

$\max u = \log(A) + \log(X)$, subject to $A + wX \leq M, A \geq 0, X \geq 0$
Variables: A, X
parameters: w, M
Conditions on parameters: $w > 0, M > 0$

The solutions are

$$(A, X) = \left(\frac{M}{2}, \frac{M}{2w}\right) \tag{29}$$

4.1. SOLVE THE OPTIMIZATION PROBLEM OF FIRM 1

$\max \Pi_1 = A_1 - wX_1 - qE_1$
subject to $X_1 \geq 0, A_1 \geq 0, A_1 = 2\sqrt{X_1}, E_1 = X_1$
Variables: A_1, X_1, E_1
parameters: w, q
conditions on parameters: $w > 0, q > 0$

The solution is

$$(A_1, X_1, E_1, \Pi_1) = \left(\frac{2}{w+q}, \frac{1}{(w+q)^2}, \frac{1}{(w+q)^2}, \frac{1}{w+q}\right) \tag{30}$$

4.2. SOLVE THE OPTIMIZATION PROBLEM OF FIRM 2

$\max \Pi_2 = A_2 - wX_2 + qE_2$
subject to $A_2 \geq 0, X_2 \geq 0, A_2 = \dfrac{2X_2}{1+X_1}, E_2 = X_1$
Variables: A_2, X_2, E_2, X_1
parameters: w, q
conditions on parameters: $w > 0, q > 0$

There is no solution to this problem, because firm 2 can set $X_2 = 0$, and then the profit function becomes $\Pi_2 = qE_2$, which has no global maximum for any $q > 0$. Hence no equilibrium exists with $q > 0$. On the other hand, if $q = 0$, we get the same equilibria as in (13) and (14), and no correction of inefficiency is possible.

5.COMPETITIVE EQUILIBRIA WITH PRICE CONTROLS

We impose $w \geq \psi$ .When $\bar{X} \leq 3$,an equilibrium exists iff $\psi \leq \sqrt{\frac{3}{\bar{X}}}$,and it is EQ1.When $\bar{X} > 3$, if $\psi \leq \sqrt{\frac{3}{\bar{X}}}$ then both EQ1 and EQ2 are equilibria; and if $\sqrt{\frac{3}{\bar{X}}} < \psi \leq 1$ then only EQ2 is an equilibrium(the pareto inferior equilibrium EQ1 is now illegal).

PROBLEM 2

THE ECONOMY

- Three goods, A, B, X, written in this order.
- Two consumers,1 and 2.
- Two firms, α and β .

Consumer 1

- Consumption set R_+^2
- Endowment vector $\omega_1 = \overbrace{[0,0,60]}^{\text{A,B,X}}$
- Profit shares $\theta_{1\alpha} = 0 = \theta_{1\beta}$
- Utility function $u_1(A_1, B_1) = A_1 + B_1$

Consumer 2

- Consumption set R_+
- Endowment vector $\omega_2 = \overbrace{[0,0,0]}^{\text{A,B,X}}$
- Profit share $\theta_{2\alpha} = 1 = \theta_{2\beta}$
- Utility function $u_2(X_2) = X_2$

firm α produces good A out of good X with technology described by the production function $A = \sqrt{2X_\alpha}$

firm β produces good B out of good X with technology described by the production function $B = \sqrt{2X_\beta}$

The pareto problem of this economy

objective functions

$u_1 = A_1 + B_1, u_2 = X_2$

Feasible set

$S = \left\{ z \in \mathbb{R}_+^7 : X_2 + X_\alpha + X_\beta \le 60, A_1 \le A, B_1 \le B, A = \sqrt{2X_\alpha}, B = \sqrt{2X_\beta} \right\}$

for z=$(A, B, A_1, B_1, X_2, X_\alpha, X_\beta)$ (31)

QUESTIONS

(Some of the answers might depend on the values of the parameters)

1. Compute all efficient points of this economy, i.e., solve the vector maximization problem (31).

2. Suppose that firm α pays a tax $0 \le t < 1$ per unit of revenue; tax proceeds are given to consumer 2 with a lump-sum transfer. Compute competitive equilibria as a function of the tax rate t, using the normalization rule "price of good X=1". Let $v_1(t), v_2(t)$ be the values of the utility functions u_1, u_2 in equilibrium, and $z(t)=(A(t), B(t), A_1(t), B_1(t), X_2(t), X_\alpha(t), X_\beta(t))$ the equilibrium allocation.

3. Compute, if it exists, the tax rate t_0 that maximizes the sum of equilibrium utilities $W(t) = v_1(t) + v_2(t)$ subject to the constraint $0 \le t < 1$. Is the allocation $z(t_0)$ efficient? If your answer is no, define and compute a measure of the inefficiency.

4. Compute, if it exists, the tax rate t_R that maximizes equilibrium tax proceeds subject to the constraint $0 \le t < 1$. Is the allocation $z(t_R)$ efficient? If your answer is no, define and compute a measure of the inefficiency.

5. Compute, if it exists, the tax rate t_c that maximizes the equilibrium value of consumption $vc(t) = p_A(t)A_1(t) + p_B(t)B_1(t) + X_2(t)$ subject to the constraint $0 \le t < 1$. Is the allocation $z(t_c)$ efficient? If your answer is no, define and compute a measure of the inefficiency.

ANSWERS TO PROBLEM 2

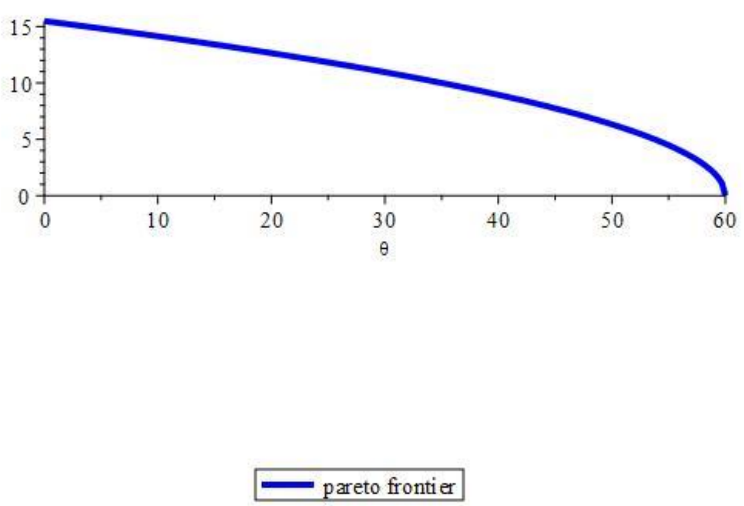
1.EFFICIENT POINTS

Efficient points are the global maxima of the following maximization problem

$$\boxed{\begin{array}{l}\text{objective function } u_1 = A_1 + B_1 \\ \text{constraints} \\ X_2 \geq \theta, X_2 + X_\alpha + X_\beta \leq 60, A_1 \leq A, B_1 \leq B, A = \sqrt{2X_\alpha}, B = \sqrt{2X_\beta} \\ (A, B, A_1, B_1, X_2, X_\alpha, X_\beta) \in \mathbb{R}^7_+ \\ \text{variables:} A, B, A_1, B_1, X_2, X_\alpha, X_\beta \\ \text{parameters:} \theta \in \mathbb{R}\end{array}} \tag{32}$$

The solutions are, after eliminating θ

$$\boxed{\begin{array}{l}\text{efficient points} \\ X_\alpha = X_\beta = 30 - \dfrac{X_2}{2}, A = B = A_1 = B_1 = \sqrt{60 - X_2}, 0 \leq X_2 \leq 60 \\ \text{pareto frontier} \\ u_1 = 2\sqrt{60 - u_2}, 0 \leq u_2 \leq 60\end{array}} \tag{33}$$



2.COMPETITIVE EQUILIBRIA WITH TAXATION

1.NAME THE PRICE OF EACH GOOD

$p_A =$ price of commodity A,$p_B =$ price of commodity B
w=price of commodity X.Normalize w $=1$
T=lump-sum transfer to the consumer

2. DEFINE CONSUMER INCOMES

$$M_1 = 60, M_2 = \Pi_\alpha + \Pi_\beta + T \tag{34}$$

3.1. SOLVE THE OPTIMIZATION PROBLEM OF CONSUMER 1

max $u_1 = A_1 + B_1$,subject to $p_A A_1 + p_B B_1 \le M_1, A_1 \ge 0, B_1 \ge 0$
Variables: A_1, B_1
parameters: p_A, p_B, M_1
Conditions on parameters: $p_A > 0, p_B > 0, M_1 > 0$

The solutions are

$$(A_1, B_1) = \begin{cases} \left(0, \dfrac{M_1}{p_B}\right) & \text{if} \quad p_B < p_A \\ \left(A_1, \dfrac{M_1}{p_B} - A_1\right), 0 \le A_1 \le \dfrac{M_1}{p_B} & \text{if} \quad p_B = p_A \\ \left(\dfrac{M_1}{p_A}, 0\right) & \text{if} \quad p_B > p_A \end{cases} \tag{35}$$

3.2. SOLVE THE OPTIMIZATION PROBLEM OF CONSUMER 2

max $u_2 = X_2$,subject to $X_2 \le M_2, X_2 \ge 0$
Variables: X_2
parameters: M_2
Conditions on parameters: $M_2 > 0$

The solutions are

$$X_2 = M_2 \tag{36}$$

4.A. SOLVE THE OPTIMIZATION PROBLEM OF FIRM α

max $\Pi_\alpha = (1-t)\, p_A A - X_\alpha$

subject to $X_\alpha \geq 0, A \geq 0, A = \sqrt{2X_\alpha}$

Variables:A, X_α

parameters: p_A

conditions on parameters:$p_A > 0, 0 \leq t < 1$

The solution is

$$\left(A, X_\alpha, \Pi_\alpha\right) = \left((1-t)\, p_A, \frac{(1-t)^2 {p_A}^2}{2}, \frac{(1-t)^2 {p_A}^2}{2} \right) \tag{37}$$

4.B. SOLVE THE OPTIMIZATION PROBLEM OF FIRM β

max $\Pi_\beta = p_B B - X_\beta$

subject to $X_\beta \geq 0, B \geq 0, B = \sqrt{2X_\beta}$

Variables:B, X_β

parameters: p_B

conditions on parameters:$p_B > 0$

The solution is

$$\left(B, X_\beta, \Pi_\beta\right) = \left(p_B, \frac{{p_B}^2}{2}, \frac{{p_B}^2}{2} \right) \tag{38}$$

5. SOLVE THE EQUILIBRIUM CONDITIONS

$$\boxed{A = A_1, B = B_1, X_\alpha + X_\beta + X_2 = 60} \tag{39}$$

$$\begin{array}{l}
\boxed{\text{Competitive equilibrium } z(t)} \\
p_A = p_B = 2\sqrt{\dfrac{15}{2-t}} \\
(A, X_\alpha, \Pi_\alpha) = \left(\dfrac{2\sqrt{15}(1-t)}{\sqrt{2-t}}, \dfrac{30(1-t)^2}{2-t}, \dfrac{30(1-t)^2}{2-t}\right) \\
(B, X_\beta, \Pi_\beta) = \left(\dfrac{2\sqrt{15}}{\sqrt{2-t}}, \dfrac{30}{2-t}, \dfrac{30}{2-t}\right) \\
(A_1, B_1) = \left(\dfrac{2\sqrt{15}(1-t)}{\sqrt{2-t}}, \dfrac{2\sqrt{15}}{\sqrt{2-t}}\right), X_2 = \dfrac{30(2-t^2)}{2-t} \\
v_1(t) = 2\sqrt{15}\sqrt{2-t}, v_2(t) = \dfrac{30(2-t^2)}{2-t} \\
T(t) = \dfrac{60(1-t)t}{2-t}
\end{array} \tag{40}$$

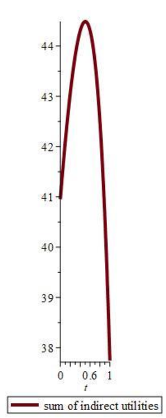
Comparing (40) to (33) we conclude that

$$\begin{array}{l}
\boxed{z(t) \text{ is efficient iff } t = 0} \\
\boxed{v_1(0) = 2\sqrt{2}\sqrt{15}, v_2(0) = 30}
\end{array} \tag{41}$$

3.MAX SUM OF INDIRECT UTILITIES

We solve the following max problem

$$\max W(t) = v_1(t) + v_2(t) = 2\sqrt{15}\sqrt{2-t} + \frac{30(2-t^2)}{2-t}, \text{subject to } 0 \le t \le 1, \text{variable:} t \tag{42}$$



The solution is $\boxed{t_0 = 0.486421034}$ .The corresponding competitive equilibrium is

$$\boxed{\begin{array}{l} \boxed{\text{competitive equilibrium } z(t_0)} \\ p_A = p_B = 1.625651518\sqrt{15} \\ (A, X_\alpha, \Pi_\alpha) = (0.8349004258\sqrt{15}, 5.227940403, 5.227940403) \\ (B, X_\beta, \Pi_\beta) = (1.625651518\sqrt{15}, 19.82057142, 19.82057142) \\ (A_1, B_1) = \left(0.8349004258\sqrt{15}, 1.625651518\sqrt{15}\right), X_2 = 34.95148817 \\ v_1(t_0) = 2.460551942\sqrt{15}, v_2(t_0) = 34.95148817 \end{array}} \tag{43}$$

By (41) and (33), $z(t_0)$ is inefficient.

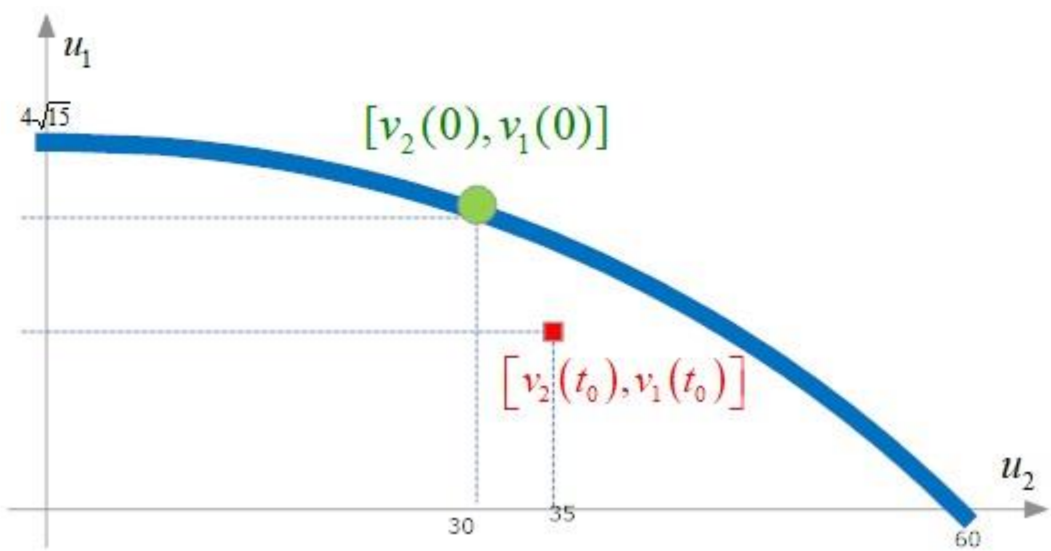


Figure 10 position of equilibrium utilities relative to the pareto frontier

We conclude that choosing the optimal tax rate by maximizing the sum of indirect utilities is a mistake; An estimate of the mistake is the distance from the point $[v_2(t_0), v_1(t_0)]$ to the point $[v_2(t_0), 2\sqrt{60 - v_2(t_0)}]$ on the pareto frontier i.e. the number $2\sqrt{60 - v_2(t_0)} - v_1(t_0) = 1.424774457$

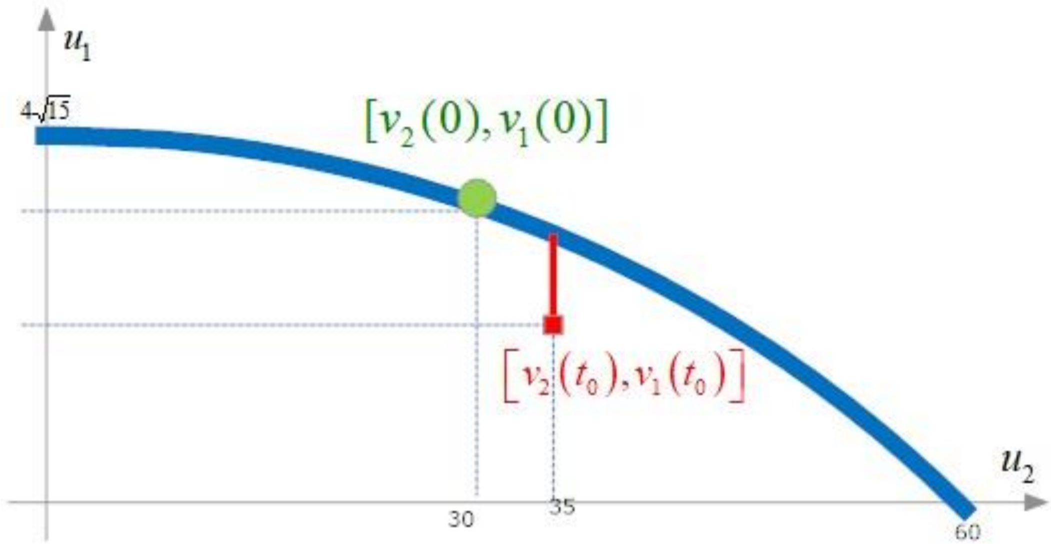
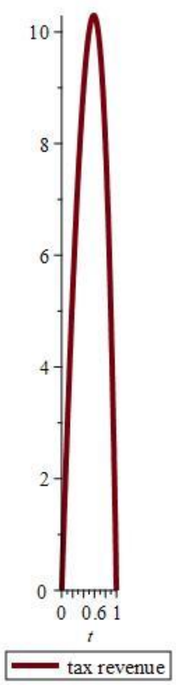


Figure 11 size of the mistake

4.MAX TAX REVENUE

We solve the following max problem

$$\max T(t) = \frac{60(1-t)t}{2-t}, \text{subject to } 0 \le t \le 1, \text{variable:} t \tag{44}$$



The solution is $\boxed{t_R = 2 - \sqrt{2}}$ .The corresponding competitive equilibrium is

$$\boxed{\begin{array}{l} \boxed{\text{competitive equilibrium } z(t_R)} \\ p_A = p_B = 2^{3/4}\sqrt{15} \\ \left(A, X_\alpha, \Pi_\alpha\right) = (\sqrt{15}\left(\sqrt{2}-1\right)2^{3/4}, -60+45\sqrt{2}, -60+45\sqrt{2}) \\ \left(B, X_\beta, \Pi_\beta\right) = (2^{3/4}\sqrt{15}, 15\sqrt{2}, 15\sqrt{2}) \\ \left(A_1, B_1\right) = \left(\sqrt{15}\left(\sqrt{2}-1\right)2^{3/4}, 2^{3/4}\sqrt{15}\right), X_2 = 120 - 60\sqrt{2} \\ v_1\left(t_R\right) = 2^{5/4}\sqrt{15}, v_2\left(t_R\right) = 120 - 60\sqrt{2} \end{array}} \tag{45}$$

By (45) and (33), $z\left(t_R\right)$ is inefficient. We conclude that choosing the optimal tax rate by maximizing tax revenue is a mistake; An estimate of the mistake is the distance from the point $\left[v_2\left(t_R\right), v_1\left(t_R\right)\right]$ to the point $[v_2\left(t_R\right), 2\sqrt{60 - v_2\left(t_R\right)}]$ on the pareto frontier i.e. the number $2\sqrt{60 - v_2\left(t_R\right)} - v_1\left(t_R\right) = 2^{5/4}\sqrt{15}(2^{1/4} - 1)$

5.MAX CONSUMPTION VALUE

We solve the following max problem

$$\begin{array}{l} \max\ vc(t) = p_A(t)A_1\left(t\right) + p_B(t)B_1\left(t\right) + X_2\left(t\right) = \dfrac{30\left(t^2 + 2t - 6\right)}{t-2} \\ \text{subject to } 0 \le t \le 1 \\ \text{variable:} t \end{array} \tag{46}$$

The solution is $\boxed{t_c = t_R = 2 - \sqrt{2}}$ .The corresponding competitive equilibrium is (45)

PROBLEM 3

Let A, B be two consumers about whom we have the price-quantity datasets $F_A = [(p^1, A^1), (p^2, A^2)], F_B = [(p^1, B^1), (p^2, B^2)]$, where

$$\begin{array}{l} p^1 = [8,\ 2,\ 2],\ p^2 = [1, 3, 2] \\ A^1 = [3, 2, 0],\ A^2 = [2, 0, 0] \\ B^1 = [0, \theta, 1],\ B^2 = [0, 2, 2], \theta \ge 0 \end{array} \tag{47}$$

Let $F = [(p^1, A^1 + B^1), (p^2, A^2 + B^2)]$

1.Compute the set

$$\Theta = \{\theta \geq 0, \text{the datasets } F_A, F_B \text{ both satisfy GARP}\} \tag{48}$$

2. Compute the set

$$\mathrm{H} = \{\theta \geq 0, \text{the dataset } F \text{ fails GARP}\} \tag{49}$$

3.Compute the set $\Theta \bigcap \mathrm{H}$

4.Suppose $\theta = \frac{1}{4}$ .Does there exist a positive representative consumer?

5.Compute the Afriat utility functions of consumers A, B for $\theta = \frac{1}{4}$

ANSWERS TO PROBLEM 3

CONSUMER A DATA SATISFY GARP

$$\text{Afriatmatrix[A]} := \begin{bmatrix} 0 & -12 \\ 7 & 0 \end{bmatrix} \tag{50}$$

Figure 12 Revealed preference graph of consumer A

The data from consumer A satisfy GARP

CONSUMER B DATA SATISFY GARP

$$\text{Afriatmatrix[B]} := \begin{bmatrix} 0 & 6-2\theta \\ 3\theta-8 & 0 \end{bmatrix} \tag{51}$$

The data from consumer B satisfy GARP because the system of inequalities $[6-2\theta \leq 0, 3\theta-8 \leq 0, \theta \geq 0]$ is not satisfiable. Hence

$$\boxed{\Theta = \mathbb{R}_+} \tag{52}$$

AGGREGATE DATA FAIL GARP FOR $0 \le \theta \le 1/3$

$$\text{Afriatmatrix[F]:=}\begin{bmatrix} 0 & -6-2\theta \\ 3\theta-1 & 0 \end{bmatrix} \tag{53}$$

$$H = [0,1/3] = H \cap \Theta \tag{54}$$

No positive representative consumer exists for $0 \le \theta \le 1/3$.

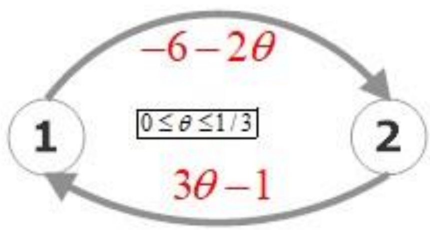


Figure 13 Revealed preference graph induced by F

For $\theta = \frac{1}{4}$ the Afriat utility functions are

$$\begin{aligned} u_A &= \min\left(-15 + 8x_1 + 2x_2 + 2x_3, -\frac{17}{7} + \frac{12x_1}{7} + \frac{36x_2}{7} + \frac{24x_3}{7}\right) \\ u_B &= \min\left(-\frac{101}{44} + \frac{116x_1}{11} + \frac{29x_2}{11} + \frac{29x_3}{11}, -\frac{7}{4} + x_1 + 3x_2 + 2x_3\right) \end{aligned} \tag{55}$$