Question 1

Consider an economy with N consumers, two firms, and two goods.

- Goods: A,B
- Firm 1 produces goods A using good B as an input, with a technology described by the production function

$$A_1 = F(B_1) \tag{1}$$

F exhibits constant returns to scale and satisfies $F(25) = 5$

- Firm 2 produces goods A using good B as an input, with a technology described by the production function

$$A_2 = 2\sqrt{B_2} \tag{2}$$

- The aggregate consumer demand function $A(p,w)$ for good A satisfies

$$A(16,4) = 8, A(75,15) < 9 \tag{3}$$

where p is the price of good A is and w is the price of good B.

Compute at least one competitive equilibrium price vector of this economy, and the corresponding equilibrium quantities for firms.

ANSWER 1

1. The production function F satisfies $F(B_1) = B_1 F(1)$,hence $5 = F(25) = 25F(1)$, and

$$F(B_1) = \frac{B_1}{5} \tag{4}$$

The profit function of firm 1 is then $\Pi_1 = \left(\frac{p}{5} - w\right) B_1$; it is maximized at

$$(B_1, A_1, \Pi_1) = \begin{cases} (\infty, \infty, \infty) & \text{if} \quad p > 5w \\ \left(B_1, \frac{B_1}{5}, 0\right) & \text{if} \quad p = 5w \\ (0,0,0) & \text{if} \quad p < 5w \end{cases} \tag{5}$$

2. The profit function of firm 2 is $\Pi_2 = 2p\sqrt{B_2} - wB_2$;it is maximized at

$$\left(B_2, A_2, \Pi_2\right) = \left(\left(\frac{p}{w}\right)^2, 2\frac{p}{w}, \frac{p^2}{w}\right) \tag{6}$$

3. The equilibrium conditions are

$$A\left(p,w\right) = A_1(\text{p},\text{w}) + A_2(\text{p},\text{w}) \tag{7}$$

If $p = 5w$ is an equilibrium then(5),(6),(7) imply $A\left(5\text{w},w\right) = A_1(5w,\text{w}) + 10$,hence by the homogeneity of degree zero of consumer demand functions

$9 > F(75,15) = F(5,1) = F(5w,\text{w}) = A_1(5w,\text{w}) + 10 \geq 10$, a contradiction. Hence the search for equilibria can be restricted to prices that satisfy $p < 5w, A_1 = B_1 = 0$ .Then(6),(7) yield

$A\left(p,w\right) = 2\frac{p}{w}$ .We conclude from (3) and from $A\left(16,4\right) = 8 = 2\frac{16}{4}$ that an equilibrium is

$$\begin{aligned} &\frac{p}{w} = 4 \\ &A_1 = B_1 = \Pi_1 = 0, \\ &B_2 = 16, A_2 = A = 8, \Pi_2 = 4p \end{aligned} \tag{8}$$

Question 2

Consider an economy with one consumer, m+n firms with $m \geq 2, n \geq 2$, and two goods.

- Goods: A,X
- Each firm $i = 1..m$ produces good A using good X as an input, with a technology described by the production function

$$A_i = \begin{cases} 0 & \text{if} \quad X_i \leq 1 \\ 2X_i - 2 & \text{if} \quad X_i \geq 1 \end{cases} \tag{9}$$

- Each firm $i = m + 1..\text{n}$ produces good A using good X as an input, with a technology described by the production function

$$A_i = X_i \tag{10}$$

- Consumer preferences are described by a utility function $U(A,X)$ that satisfies

$$U_A > 0, U_X > 0, 1 < \frac{U_X}{U_A} \tag{11}$$

  for all feasible values of the variables A,X

- The consumer's initial endowment is four units of good X.

Compute which firms will produce, and which will remain inactive, at any Pareto efficient point of this economy

ANSWER 2

Given the assumptions on preferences, efficient points are the solutions of the following optimization problem

$$\begin{aligned}
&\max U(A,X),\text{subject to}\\
&A=\sum_{i=1}^{k}A_i+\sum_{i=m+1}^{n}A_i,\ \sum_{i=m+1}^{n}A_i+k+\frac{1}{2}\sum_{i=1}^{k}A_i+X=4,\\
&k\text{ integer},0\le k\le m,\text{all variables nonnegative}\\
&\text{Variables: }k,A,X,A_1,\ldots,A_k,A_{m+1},\ldots,A_n
\end{aligned}\tag{12}$$

where k is the number of firms with IRS technology that are active.

Given the assumptions on preferences, if $\left(\overline{k},\overline{A},\overline{X},\overline{A}_1,\ldots,\overline{A}_k,\overline{A}_{m+1},\ldots,\overline{A}_n\right)$ is an efficient point, then $\left(\overline{k},\overline{X},\overline{A}_1,\ldots,\overline{A}_k,\overline{A}_{m+1},\ldots,\overline{A}_n\right)$ is also a solution of the following minimization problem

$$\begin{aligned}
&\min \sum_{i=m+1}^{n}A_i+k+\frac{1}{2}\sum_{i=1}^{k}A_i,\text{subject to}\\
&\sum_{i=1}^{k}A_i+\sum_{i=m+1}^{n}A_i\ge\overline{A}\\
&k\text{ integer},0\le k\le m,\text{all variables nonnegative}\\
&\text{Variables: }k,A_1,\ldots,A_k,A_{m+1},\ldots,A_n\\
&\text{Parameters:}\overline{A}
\end{aligned}\tag{13}$$

namely the input requirements of producing $\overline{A}$ should be minimized.

Any solution of (13) has the following properties

$$\text{only the total output } C=\sum_{i=m+1}^{n}A_i\text{ of CRS firms matters}\tag{14}$$

$$\text{if } k\ge 1\text{ then }k=1\text{ and }C=0\tag{15}$$

Hence to solve (13) we solve the following two minimization problems, and then choose the solution that minimizes the objective function of (13)

$$\begin{aligned}
&\min 1+\frac{1}{2}A_1,\text{subject to}\\
&A_1\ge\overline{A}\\
&\text{Variables: }A_1\ge 0\\
&\text{Parameters:}\overline{A}
\end{aligned}\tag{16}$$

$$\begin{aligned}&\min C\text{, subject to}\\&C \geq \overline{A}\\&\text{Variables: } C \geq 0\\&\text{Parameters:} \overline{A}\end{aligned} \tag{17}$$

Comparing the solutions of(16),(17) we obtain the following characterization of efficient points $\left(k, A, X, A_1, \ldots, A_k, A_{m+1}, \ldots, A_n\right)$

$$\begin{aligned}&\text{if } A > 2 \text{ then } k = 1, A_1 = A, C = 0\\&\text{if } A < 2 \text{ then } k = 0, C = A\\&\text{if } A = 2 \text{ then both are solutions}\end{aligned} \tag{18}$$

We will now show that there is no efficient point with $0 < A \leq 2$ .For if such a point is efficient then it is a solution of

$$\begin{aligned}&\max U(A, X)\text{, subject to}\\&A + X \leq 4\\&\text{Variables: } A \geq 0, X \geq 0\end{aligned} \tag{19}$$

FOCS for (19) then yield $\frac{U_x}{U_A} \leq 1$,contradicting (11).Hence at any efficient point

$$A > 0 \text{ implies } \mathrm{A}_1 = A > 0, \mathrm{A}_2 = \ldots = A_{m+n} = 0 \tag{20}$$

Condition (11) shows that at any efficient point $A > 0$ .Hence at any efficient point

$$\boxed{\mathrm{A}_1 = A > 0, \mathrm{A}_2 = \ldots = A_{m+n} = 0} \tag{21}$$

and

$$\text{either A=6,X=0,} \frac{U_X(6,0)}{U_A(6,0)} \leq 2\text{, or } \frac{U_X}{U_A} = 2 \text{ ,X>0,0<A<6} \tag{22}$$