

Players:  $A, B$

Set of outcomes  $X = \{\delta, \epsilon\}$

Set of possible strict preferences on  $X$   
 $U = \{\delta \succ, \epsilon \succ\}$

Notation: We represent the preference relation  $\delta \succ$  by the utility function  
 $\Delta = \{\delta, \epsilon\} \rightarrow \mathbb{R}, \Delta(\delta) = 1, \Delta(\epsilon) = 0$ . Similarly,  $\epsilon \succ$   
 is represented by  $E(\delta) = 0, E(\epsilon) = 1$ . Hence  
 $U = \{\Delta, E\}$ .

Universal domain  $U_A = U_B = U$  (UD)

Mechanism

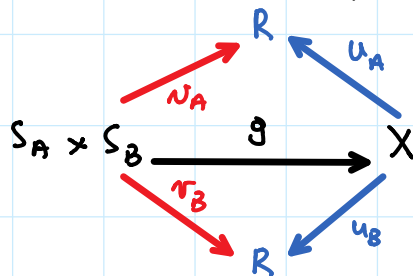
$$S_A \times S_B \xrightarrow{g} X$$

Game (of complete information) induced by the mechanism and each pair

$(u_A, u_B)$  of utility functions

$$u_A(\alpha, \beta) = u_A(g(\alpha, \beta)), \alpha \in S_A$$

$$u_B(\alpha, \beta) = u_B(g(\alpha, \beta)), \beta \in S_B$$



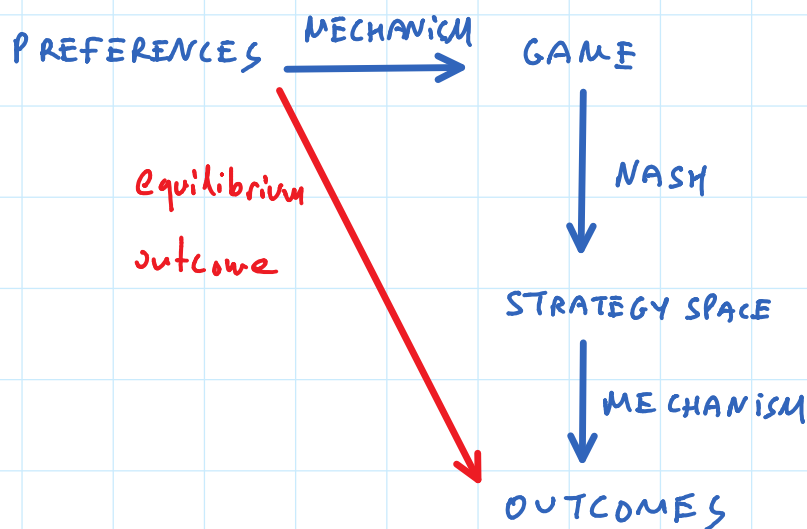
An equilibrium of this game is a pair  $(\alpha, \beta)$  of choices such that

$$\begin{aligned}
 u_A(\alpha, \beta) &\geq u_A(\alpha', \beta) && \forall \alpha' \in S_A \\
 u_B(\alpha, \beta) &\geq u_B(\alpha, \beta') && \forall \beta' \in S_B
 \end{aligned}$$

, i.e.

$$u_A g(\alpha, \beta) \geq u_A g(\alpha', \beta) \quad \forall \alpha' \in S_A \quad (1)$$

$$u_B g(\alpha, \beta) \geq u_B g(\alpha, \beta') \quad \forall \beta' \in S_B \quad (2)$$



A mechanism  $\langle S_A, S_B, g \rangle$  makes A into a dictator if for any preferences  $(u_A, u_B)$ , there exists an action  $\alpha \in S_A$  such that  $g(\alpha, \beta) = \arg \max u_A \quad \forall \beta \in S_B$ .

A mechanism is dictatorial if it makes some agent a dictator.

A mechanism is complete if for any  $u_A \in U, u_B \in U$ .

the induced game has an equilibrium.

A mechanism is efficient if for any  $u_A \in U, u_B \in U$ , all equilibria  $(\alpha, \beta)$  of the induced game generate an efficient outcome  $g(\alpha, \beta)$  so  $\forall x \in X$

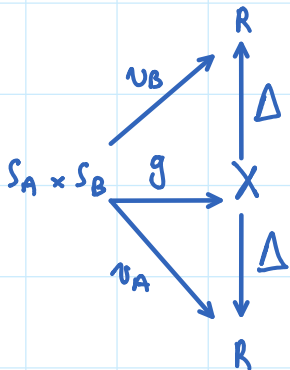
$$\begin{array}{l} u_A x \geq u_A g(\alpha, \beta) \\ u_B x \geq u_B g(\alpha, \beta) \end{array} \Bigg| \Rightarrow x = g(\alpha, \beta)$$

### HURWICZ-SCHMEIDLER THEOREM

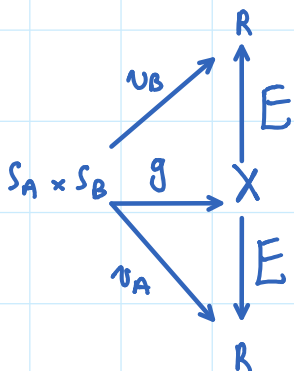
Any complete and efficient mechanism is dictatorial

Proof: Let  $\langle S_A, S_B, g \rangle$  be complete and efficient.

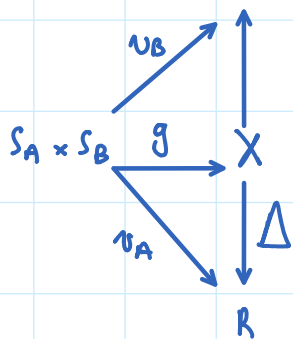
There are four games induced by this mechanism



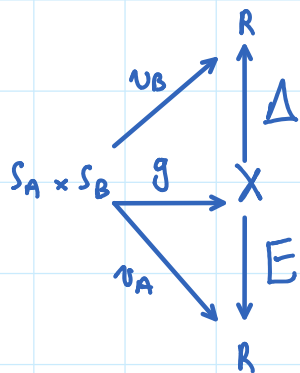
has  $\delta$  as the unique equilibrium outcome, by efficiency + completeness  $(O_\Delta)$



has  $\epsilon$  as the unique equilibrium outcome, by efficiency + completeness  $(O_E)$



has some equilibrium  $(\alpha, \beta) \in S_A \times S_B$   
by completeness



has some equilibrium  $(x, y) \in S_A \times S_B$   
by completeness

We show that

$g(\alpha, \beta) = \delta$  implies A is a dictator

$g(\alpha, \beta) = \epsilon$  implies B is a dictator.

Case 1  $g(\alpha, \beta) = \delta$

Then by (2), the equilibrium condition for player B,

$$E g(\alpha, \beta) \geq E g(\alpha, \beta') \quad \forall \beta' \in S_B$$

Since  $g(\alpha, \beta) = \delta$  we obtain

$$0 = E\delta \geq E g(\alpha, \beta') \quad \forall \beta' \in S_B.$$

This in turn implies, because  $E\epsilon = 1$ , that

This in turn implies, because  $E \in I$ , that

$$g(\alpha, \beta') = \delta \quad \forall \beta' \in S_B \quad (\text{FORCE}_A^\delta)$$

ie that agent A can force outcome  $\delta$  by choosing  $\alpha$ .

We now show that  $g(x, y) = \epsilon$

Assume for contradiction that  $g(x, y) = \delta$ . Then the equilibrium condition (1) for agent A implies

$$Eg(x, y) \geq Eg(x', y) \quad \forall x' \in S_A. \quad \text{Hence}$$

$$0 = E\delta \geq Eg(x', y) \quad \forall x' \in S_A. \quad \text{Hence}$$

$$g(x', y) = \delta \quad \forall x' \in S_A \quad (\text{FORCE}_B^\delta)$$

Agent B can force outcome  $\delta$  by choosing  $y$ .

By  $(\text{FORCE}_A^\delta)$  and  $(\text{FORCE}_B^\delta)$ ,  $(\alpha, y)$  is an equilibrium for any preferences  $(u_A, u_B)$ , since the outcome at  $(\alpha, y)$  is  $\delta$ , and the outcome of any unilateral deviation is still  $\delta$ .

In particular,  $(\alpha, y)$  is an equilibrium when  $u_A = E = u_B$ , i.e.  $\delta = g(\alpha, y)$  is an equilibrium outcome when both players prefer  $\epsilon$ , contradicting (OE). Hence

$$g(x, y) = e \quad (\text{NASH } e_A)$$

The equilibrium condition (2) for agent B then implies,

$$0 = \Delta e = \Delta g(x, y) \geq \Delta g(x, y') \quad \forall y' \in S_B. \text{ Hence}$$

$$g(x, y') = e \quad \forall y' \in S_B \quad (\text{FORCE}_A^e)$$

Agent A can force outcome  $e$  by choosing action  $x$ .

By  $\text{FORCE}_A^\delta$  and  $\text{FORCE}_A^e$ , A is a dictator, because

$$\Delta \Delta \xrightarrow{\alpha \text{ forces}} \delta = \text{argmax}_x U_A$$

$$\Delta E \xrightarrow{\alpha \text{ forces}} \delta = \text{argmax}_x U_A$$

$$E \Delta \xrightarrow{x \text{ forces}} e = \text{argmax}_x U_A$$

$$E E \xrightarrow{x \text{ forces}} e = \text{argmax}_x U_A$$

Case 2:  $g(\alpha, \beta) = e$

A similar argument shows B to be a dictator