Nash implementation and the Hurwicz-Schmeidler theorem

Players: A, B

Set of outcomes $X = \{\delta, \epsilon\}$

Set of possible strict preferences on X

$U = \{\delta\epsilon, \epsilon\delta\}$

Notation: We represent the preference relation $\delta\epsilon$ by the utility function $\Delta : \{\delta, \epsilon\} \to \mathbb{R}$, $\Delta(\delta) = 1$, $\Delta(\epsilon) = 0$. Similarly, $\epsilon\delta$ is represented by $E(\delta) = 0$, $E(\epsilon) = 1$. Hence

$U = \{\Delta, E\}$.

<u>Universal domain</u> $\qquad U_A = U_B = U \qquad$ (UD)

$\boxed{\text{Mechanism}}$ $\qquad S_A \times S_B \xrightarrow{g} X$

Game (of complete information) induced by the mechanism and each pair (u_A, u_B) of utility functions

$$v_A(\alpha, \beta) = u_A(g(\alpha, \beta)), \quad \alpha \in S_A$$

$$v_B(\alpha, \beta) = u_B(g(\alpha, \beta)), \quad \beta \in S_B$$

$$\begin{tikzcd}
 & \mathbb{R} & \\
S_A \times S_B \arrow[ur, "v_A"] \arrow[rr, "g"] \arrow[dr, "v_B"'] & & X \arrow[ul, "u_A"'] \arrow[dl, "u_B"] \\
 & \mathbb{R} &
\end{tikzcd}$$

An equilibrium of this game is a pair (α, β) of choices such that

$$v_A(\alpha, \beta) \geq v_A(\alpha', \beta) \quad \forall \alpha' \in S_A$$
$$v_B(\alpha, \beta) \geq v_B(\alpha, \beta') \quad \forall \beta' \in S_B$$

, i.e.

$$\boxed{\begin{aligned} u_A\, g(\alpha, \beta) &\geq u_A\, g(\alpha', \beta) \quad \forall \alpha' \in S_A \qquad (1) \\ u_B\, g(\alpha, \beta) &\geq u_B\, g(\alpha, \beta') \quad \forall \beta' \in S_B \qquad (2) \end{aligned}}$$

$$\begin{tikzcd}
\text{PREFERENCES} \arrow[r, "\text{MECHANISM}"] \arrow[ddr, "\text{equilibrium outcome}"'] & \text{GAME} \arrow[d, "\text{NASH}"] \\
& \text{STRATEGY SPACE} \arrow[d, "\text{MECHANISM}"] \\
& \text{OUTCOMES}
\end{tikzcd}$$

A mechanism $\langle S_A, S_B, g \rangle$ makes A into a dictator if for any preferences (u_A, u_B), there exists an action $\alpha \in S_A$ such that $g(\alpha, \beta) = \arg\max u_A \quad \forall \beta \in S_B$.

A mechanism is dictatorial if it makes some agent a dictator.

A mechanism is complete if for any $u_A \in U$, $u_B \in U$, the induced game has an equilibrium.

A mechanism is efficient if for any $u_A \in U$, $u_B \in U$, all equilibria (α, β) of the induced game generate an efficient outcome $g(\alpha, \beta)$ ie $\forall x \in X$

$$\left.\begin{array}{l} u_A\, x \geq u_A\, g(\alpha,\beta) \\ u_B\, x \geq u_B\, g(\alpha,\beta) \end{array}\right| \Rightarrow x = g(\alpha,\beta)$$

HURWICZ-SCHMEIDLER THEOREM

Any complete and efficient mechanism is dictatorial

Proof: Let $\langle S_A, S_B, g \rangle$ be complete and efficient.

There are four games induced by this mechanism

$$\begin{tikzcd} & R \\ S_A \times S_B \arrow[ur, "u_B"] \arrow[r, "g"] \arrow[dr, "u_A"'] & X \arrow[u, "\Delta"'] \arrow[d, "\Delta"] \\ & R \end{tikzcd}$$

has δ as the unique equilibrium outcome, by efficiency + completeness (O_Δ)

$$\begin{tikzcd} & R \\ S_A \times S_B \arrow[ur, "u_B"] \arrow[r, "g"] \arrow[dr, "u_A"'] & X \arrow[u, "E"'] \arrow[d, "E"] \\ & R \end{tikzcd}$$

has ε as the unique equilibrium outcome, by efficiency + completeness (O_E)

$$\begin{tikzcd} & {} \\ S_A \times S_B \arrow[ur, "v_B"] \arrow[r, "g"] \arrow[dr, "v_A"'] & X \arrow[u] \arrow[d, "\Delta"] \\ & R \end{tikzcd}$$

has some equilibrium $(\alpha, \beta) \in S_A \times S_B$ by completeness

$$\begin{tikzcd} & R \\ S_A \times S_B \arrow[ur, "v_B"] \arrow[r, "g"] \arrow[dr, "v_A"'] & X \arrow[u, "\Delta"'] \arrow[d, "E"] \\ & R \end{tikzcd}$$

has some equilibrium $(x, y) \in S_A \times S_B$ by completeness

We show that

$g(\alpha, \beta) = \delta$ implies A is a dictator

$g(\alpha, \beta) = \epsilon$ implies B is a dictator.

Case 1 $g(\alpha, \beta) = \delta$

Then by (2), the equilibrium condition for player B,

$$E\, g(\alpha, \beta) \geq E\, g(\alpha, \beta') \qquad \forall \beta' \in S_B$$

Since $g(\alpha, \beta) = \delta$ we obtain

$0 = E\delta \geq E\, g(\alpha, \beta') \qquad \forall \beta' \in S_B$.

This in turn implies, because $E\epsilon = 1$, that

This in turn implies, because $E\epsilon = 1$, that

$$g(\alpha, \beta') = \delta \qquad \forall\, \beta' \in S_B \qquad (\text{FORCE}_A^{\delta})$$

ie that agent A can force outcome δ by choosing α.

We now show that $g(x,y) = \epsilon$

Assume for contradiction that $g(x,y) = \delta$. Then the equilibrium condition (1) for agent A implies

$$E g(x, y) \geq E g(x', y) \qquad \forall\, x' \in S_A . \quad \text{Hence}$$

$$0 = E\delta \geq E g(x', y) \qquad \forall\, x' \in S_A . \quad \text{Hence}$$

$$g(x', y) = \delta \qquad \forall\, x' \in S_A \qquad (\text{FORCE}_B^{\delta})$$

Agent B can force outcome δ by choosing y.

By $(\text{FORCE}_A^{\delta})$ and $(\text{FORCE}_B^{\delta})$, (α, y) is an equilibrium for any preferences (u_A, u_B), since the outcome at (α, y) is δ, and the outcome of any unilateral deviation is still δ.

In particular, (α, y) is an equilibrium when $u_A = E = u_B$, i.e. $\delta = g(\alpha, y)$ is an equilibrium outcome when both players prefer ϵ, contradicting (O_E). Hence

$$g(x,y) = \epsilon \qquad (NASH_{E\Delta})$$

The equilibrium condition (2) for agent B then implies

$$0 = \Delta\epsilon = \Delta g(x,y) \geq \Delta g(x,y') \quad \forall y' \in S_B. \text{ Hence}$$

$$g(x,y') = \epsilon \quad \forall y' \in S_B \qquad (FORCE_A^{\epsilon})$$

Agent A can force outcome ϵ by choosing action x.

By $FORCE_A^{\delta}$ and $FORCE_A^{\epsilon}$, A is a dictator, because

$$\Delta\Delta \quad \xrightarrow{\alpha \text{ forces}} \quad \delta = \operatorname{argmax} u_A$$

$$\Delta E \quad \xrightarrow{\alpha \text{ forces}} \quad \delta = \operatorname{argmax} u_A$$

$$E\Delta \quad \xrightarrow{x \text{ forces}} \quad \epsilon = \operatorname{argmax} u_A$$

$$EE \quad \xrightarrow{x \text{ forces}} \quad \epsilon = \operatorname{argmax} u_A$$

Case 2: $g(\alpha,\beta) = \epsilon$

A similar argument shows B to be a dictator