Players: A, B

Set of outcomes $X = \{\kappa, \lambda, \mu\}$

Set of possible strict preferences on X

$U = \{\kappa\lambda\mu, \kappa\mu\lambda, \lambda\kappa\mu, \lambda\mu\kappa, \mu\lambda\kappa, \mu\kappa\lambda\} = \{1, 2, 3, 4, 5, 6\}$

Notation: Each preference relation xyz is represented by a utility function, also denoted by xyz, defined by $xyz(x) = 3$, $xyz(y) = 2$, $xyz(z) = 1$

<u>Universal domain</u> $U_A = U_B = U$ (UD)

<u>Social choice function</u> $f: U \times U \to X$

A social choice function assigns an outcome to each pair of preferences. It can be represented by a table, where each cell (i, j) is occupied by the outcome $f(i, j)$

A \ B	1 $\kappa\lambda\mu$	2 $\kappa\mu\lambda$	3 $\lambda\kappa\mu$	4 $\lambda\mu\kappa$	5 $\mu\kappa\lambda$	6 $\mu\lambda\kappa$
1 $\kappa\lambda\mu$						
2 $\kappa\mu\lambda$						
3 $\lambda\kappa\mu$						
4 $\lambda\mu\kappa$						
5 $\mu\kappa\lambda$						
6 $\mu\lambda\kappa$						

Definitions

We define some properties that a SCF f might have

f is <u>efficient</u> if for any preferences (u_A, u_B), the outcome $f(u_A, u_B)$ is efficient with respect to these preferences, namely

$$\left.\begin{array}{l} u_A\, x \geq u_A\, f(u_A, u_B) \\ u_B\, x \geq u_B\, f(u_A, u_B) \end{array}\right| \Rightarrow x = f(u_A, u_B) \qquad \text{(EFF)}$$

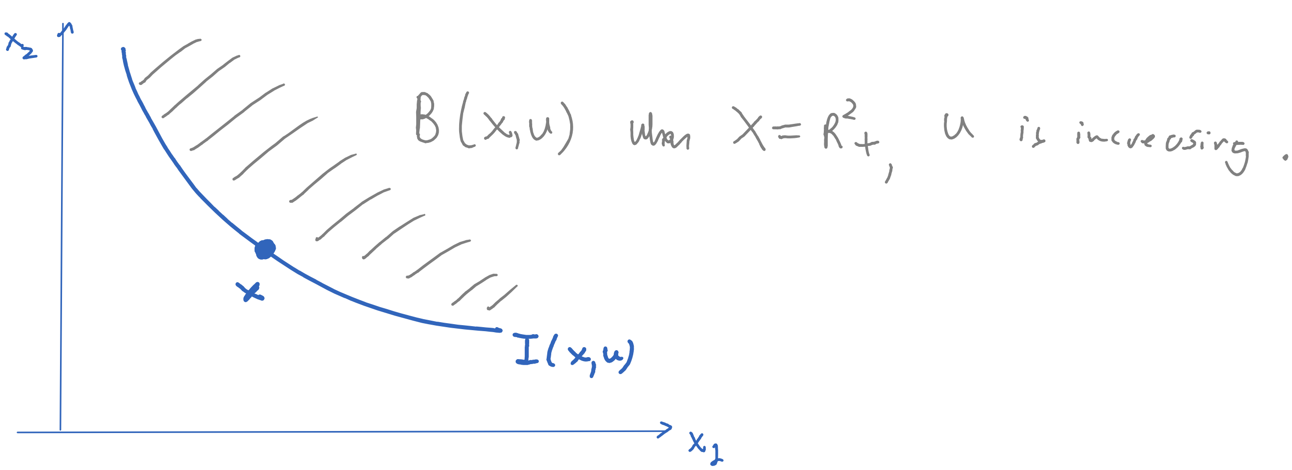
f is <u>unanimous</u> if for any preferences (u_A, u_B) that have the same maximum, it chooses this maximum, ie

$$\arg\max u_A = \arg\max u_B = x \Rightarrow f(u_A, u_B) = x \qquad \text{(UNAN)}$$

The <u>better-than set of $u: X \to R$ at a point $x \in X$</u> is the set of points that are weakly preferred to

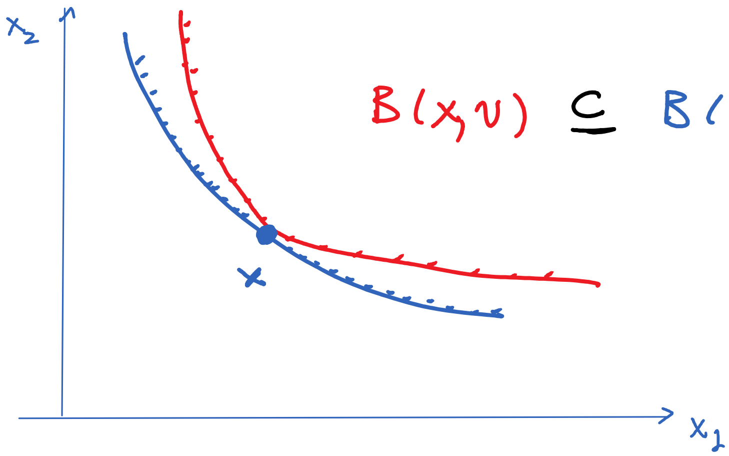
$$B(x, u) = \{ y \in X : u(y) \geq u(x) \}$$

Note that $x \in B(x, u) \quad \forall x \; \forall u$



A point x is more liked at v than at u if

$$B(x, v) \subseteq B(x, u)$$



Suppose f chooses a when preferences are (u_A, u_B), and keeps choosing a at any preferences (v_A, v_B) where a is more liked by both players. Then f is called monotonic

$$\left.\begin{array}{l} f(u_A, u_B) = a \\ B(a, v_A) \subseteq B(a, u_A) \\ B(a, v_B) \subseteq B(a, u_B) \end{array}\right| \Rightarrow f(v_A, v_B) = a \qquad \text{(MON)}$$

$\forall$ ALL

<u>f is incentive-compatible (IC)</u> if

$$u_A f(u_A, u_B) \geq u_A f(\hat{u}_A, u_B) \qquad (IC_A)$$

$$u_B f(u_A, u_B) \geq u_B f(u_A, \hat{u}_B) \qquad (IC_B)$$

$\forall$ ALL

<u>f makes player A a dictator</u> if

f makes player A a dictator if

$$f(u_A, u_B) = \arg\max u_A, \quad \forall u_A \forall u_B \qquad (DICT_A)$$

f is dictatorial if it makes some player into a dictator (DICT)

f is extreme if for any preference (u_A, u_B)

EITHER $f(u_A, u_B) = \arg\max u_A$ (EXTR)

OR $f(u_A, u_B) = \arg\max u_B$

Lemma 1: $\boxed{IC \Rightarrow MON}$

Suppose that

$$f(u_A, u_B) = a \qquad (1)$$

Show that $f(v_A, v_B) = a$

$$B(a, v_A) \subseteq B(a, u_A) \qquad (2)$$

$$B(a, v_B) \subseteq B(a, u_B) \qquad (3)$$

Let

$$f(v_A, u_B) = b \qquad (4)$$

IC implies that

$$u_A f(u_A, u_B) \geq u_A f(v_A, u_B) \qquad (5)$$

$$v_A f(v_A, u_B) \geq v_A f(u_A, u_B) \qquad (6)$$

By (1), (4), (5), (6)

$$u_A(a) \geq u_A(b) \quad (7)$$

, ie

$$v_A(b) \geq v_A(a), \text{ i.e. } b \in B(a, v_A). \quad (8)$$

By (8) and (2), $b \in B(a, u_A)$ ie

$$u_A(b) \geq u_A(a) \quad (9)$$

By (9) and (7), $a = b$ ie

$$f(u_A, u_B) = f(v_A, u_B) = a \quad (10)$$

Let $f(v_A, v_B) = c$. (11)

IC implies

$$v_B\, f(v_A, v_B) \geq v_B\, f(v_A, u_B) \quad (12)$$

$$u_B\, f(v_A, u_B) \geq u_B\, f(v_A, v_B) \quad (13)$$

By (11), (12), (13), (10)

$$v_B(c) \geq v_B(a), \text{ i.e. } c \in B(a, v_B) \quad (14)$$

$$u_B(a) \geq u_B(c) \quad (15)$$

By (14), (3), $c \in B(a, u_B)$ ie

$$u_B(c) \geq u_B(a) \quad (16)$$

By (16) (15), $c = a$. Hence by (11), (10)

$$f(v_A, v_B) = f(u_A, u_B) \qquad (17)$$

We have shown that (1), (2), (3) imply (17), ie that f is monotonic.

Lemma 2 EFF ⇒ UNAN

Let $x = \arg\max u_A = \arg\max u_B$; Hence

$$u_A(x) \geq u_A\, f(u_A, u_B) \qquad (1)$$

$$u_B(x) \geq u_B\, f(u_A, u_B) \qquad (2)$$

(1), (2) and (EFF) imply $x = f(u_A, u_B)$, ie that f is unanimous

Lemma 3 EFF + IC ⇒ EXTR

Since f is efficient and incentive-compatible, it is also monotonic and unanimous. Let

$$a = \arg\max u_A, \quad b = \arg\max u_B \qquad (1)$$

If $a = b$ UNAN implies $f(u_A, u_B) = a$, ie

$$\arg\max u_A = \arg\max u_B \Rightarrow f(u_A, u_B) = \arg\max u_A \qquad (2)$$

If $a \neq b$, let

$$c = f(u_A, u_B) \qquad (3)$$

If $c = a$ or $c = b$ then f is extreme and the lemma is proven.

Hence assume for contradiction that

If $c = a$ or $c = b$ then f is extreme and the claim is proven.

Hence assume, for contradiction, that

$$c \neq a \text{ and } c \neq b \qquad (4)$$

IC implies

$$u_B f(u_A, u_B) \geq u_B f(u_A, u_A) \qquad (5)$$

Unanimity implies $f(u_A, u_A) = a$; hence (5) implies

$u_B(c) \geq u_B(a)$, and then (4) implies

$$u_B(c) > u_B(a) \qquad (6)$$

By (6) and (1)

$$u_B = b\,c\,a \qquad (7)$$

IC implies

$$u_A f(u_A, u_B) \geq u_A f(u_B, u_B) \qquad (8)$$

Unanimity implies $f(u_B, u_B) = b$; hence (8) implies

$u_A(c) \geq u_A(b)$, and then $c \neq b$ implies

$$u_A(c) > u_A(b) \qquad (9)$$

By (9) and (1)

$$u_A = a\,c\,b \qquad (10)$$

Let $v = b\,a\,c$. Then $B(c, v) = \{a, b, c\}$, $f(v, v) = b$

$$\left.\begin{array}{l} B(c, u_A) \subset B(c, v) \\ B(c, u_B) \subset B(c, v) \\ f(v, v) = b \end{array}\right\} \qquad (11)$$

(11) and monotonicity imply $f(u_A, u_B) = b$, ie $c = b$, contradicting (4). QED.

The Gibbard -Satterthwaite theorem

If a social choice function f satisfies

- universal domain
- efficiency,and
- implementability

then it is dictatorial

Proof If f is implementable, then (by the revelation principle) it is incentive compatible. Hence f satisfies UD + EFF + IC. It follows that f satisfies UNAN + MON + EXTR.

By UNAN f satisfies

A \ B	1 $\kappa\lambda\mu$	2 $\kappa\mu\lambda$	3 $\lambda\kappa\mu$	4 $\lambda\mu\kappa$	5 $\mu\kappa\lambda$	6 $\mu\lambda\kappa$
1 $\kappa\lambda\mu$	κ	κ				
2 $\kappa\mu\lambda$	κ	κ				
3 $\lambda\kappa\mu$			λ	λ		
4 $\lambda\mu\kappa$			λ	λ		
5 $\mu\kappa\lambda$					μ	μ
6 $\mu\lambda\kappa$					μ	μ

Since f is extreme $f(1,4) = \kappa$ or $f(1,4) = \lambda$

We will show that if $f(1,4) = \kappa$ then f makes A a dictator, while $f(1,4) = \lambda$ then f makes B a dictator

Suppose $f(1,4) = \kappa$.

A \ B	1 κ∂ρ	2 κρ∂	3 ∂κρ	4 ∂ρκ	5 ρκ∂	6 ρ∂κ
1 κ∂ρ	κ	κ		κ		
2 κρ∂	κ	κ				
3 ∂κρ			∂	∂		
4 ∂ρκ			∂	∂		
5 ρκ∂					ρ	ρ
6 ρ∂κ					ρ	ρ

Note that $B(\kappa, \partial\rho\kappa) = \{\partial, \rho, \kappa\} \supset \{\rho, \kappa\} = B(\kappa, \rho\kappa\partial)$

$$\Vert$$

$B(\kappa, \rho\partial\kappa)$

Hence by monotonicity, $f(1,5) = f(1,6) = \kappa$

A \ B	1 κ∂ρ	2 κρ∂	3 ∂κρ	4 ∂ρκ	5 ρκ∂	6 ρ∂κ
1 κ∂ρ	κ	κ		κ	κ	κ
2 κρ∂	κ	κ				
3 ∂κρ			∂	∂		
4 ∂ρκ			∂	∂		
5 ρκ∂					ρ	ρ
6 ρ∂κ					ρ	ρ

By monotonicity again, $f(2,4) = f(2,5) = f(2,6) = \kappa$

A \ B	1 κ∂ρ	2 κρ∂	3 ∂κρ	4 ∂ρκ	5 ρκ∂	6 ρ∂κ
1 κ∂ρ	κ	κ	?	κ	κ	κ
2 κρ∂	κ	κ	?	κ	κ	κ

2 κμλ	κ	κ	?	κ	κ	κ
3 λκμ			λ	λ		
4 λμκ			λ	λ		
5 μκλ					μ	μ
6 μλκ					μ	μ

By extremism $f(1,3) \in \{\kappa, \lambda\}$. Then

$$3 = \lambda\mu\kappa(\kappa) = \lambda\mu\kappa\, f(1,4) = \lambda\mu\kappa\, f(\kappa\lambda\mu, \lambda\mu\kappa) \overset{IC}{\geq}$$

$$\lambda\mu\kappa\, f(\kappa\lambda\mu, \lambda\kappa\mu) = \lambda\mu\kappa\, f(1,3)$$, ie $f(1,3) = \kappa$.

Similarly $f(2,3) = \kappa$

A \ B	1 κλμ	2 κμλ	3 λκμ	4 λμκ	5 μκλ	6 μλκ
1 κλμ	κ	κ	κ	κ	κ	κ
2 κμλ	κ	κ	κ	κ	κ	κ
3 λκμ			λ	λ		?
4 λμκ			λ	λ		
5 μκλ					μ	μ
6 μλκ					μ	μ

By extremism, $f(3,6) \in \{\lambda, \mu\}$. By IC

$$\lambda\kappa\mu\, f(\lambda\kappa\mu, \mu\lambda\kappa) \geq \lambda\kappa\mu\, f(\kappa\mu\lambda, \mu\lambda\kappa) = \lambda\kappa\mu(\kappa) = 2$$. Hence

$\lambda\kappa\mu(f(3,6)) \geq 2$ ie $f(3,6) \in \{\lambda, \kappa\}$. Hence $f(3,6) = \lambda$

A \ B	1 κλμ	2 κμλ	3 λκμ	4 λμκ	5 μκλ	6 μλκ
1 κλμ	κ	κ	κ	κ	κ	κ
2 κμλ	κ	κ	κ	κ	κ	κ
3 λκμ			λ	λ		λ

		1	2	3	4	5	6
2	κμλ	κ	κ	κ	κ	κ	κ
3	λκμ			λ	λ		λ
4	λμκ			λ	λ		
5	μκλ					μ	μ
6	μλκ					μ	μ

By monotonicity

A \ B		1 κλμ	2 κμλ	3 λκμ	4 λμκ	5 μκλ	6 μλκ
1	κλμ	κ	κ	κ	κ	κ	κ
2	κμλ	κ	κ	κ	κ	κ	κ
3	λκμ			λ	λ		λ
4	λμκ			λ	λ	?	λ
5	μκλ					μ	μ
6	μλκ					μ	μ

By extremism $f(4,5) \in \{\lambda, \mu\}$. Then we have

$$\lambda = \mu\lambda\kappa(\lambda) = \mu\lambda\kappa \; f(4,6) = \mu\lambda\kappa \; f(\lambda\mu\kappa, \mu\lambda\kappa) \overset{(1c)}{\geq} \mu\lambda\kappa \; f(\lambda\mu\kappa, \mu\kappa\lambda) = \mu\lambda\kappa \; f(4,5),$$

ie $f(4,5) \in \{\lambda, \kappa\}$. Hence $f(4,5) = \lambda$.

A \ B		1 κλμ	2 κμλ	3 λκμ	4 λμκ	5 μκλ	6 μλκ
1	κλμ	κ	κ	κ	κ	κ	κ
2	κμλ	κ	κ	κ	κ	κ	κ
3	λκμ			λ	λ	?	λ
4	λμκ			λ	λ	λ	λ
5	μκλ					μ	μ
6	μλκ					μ	μ

By extremism $f(3,5) \in \{\lambda, \mu\}$. Then

$2 = \mu\lambda\kappa(\lambda) = \mu\lambda\kappa\, f(3,6) = \mu\lambda\kappa\, f(\lambda\kappa\mu, \mu\lambda\kappa) \overset{IC}{\geq}$

$\mu\lambda\kappa\, f(\lambda\kappa\mu, \mu\kappa\lambda) = \mu\lambda\kappa\, f(3,5)$, ie $f(3,5) \in \{\kappa, \lambda\}$

Hence $f(3,5) = \lambda$

A \ B	1 κλμ	2 κμλ	3 λκμ	4 λμκ	5 μκλ	6 μλκ
1 κλμ	κ	κ	κ	κ	κ	κ
2 κμλ	κ	κ	κ	κ	κ	κ
3 λκμ			λ	λ	λ	λ
4 λμκ			λ	λ	λ	λ
5 μκλ				?	μ	μ
6 μλκ					μ	μ

By extremism $f(5,4) \in \{\lambda, \mu\}$. Then by IC

$\mu\kappa\lambda\, f(5,4) = \mu\kappa\lambda\, f(\mu\kappa\lambda, \lambda\mu\kappa) \geq \mu\kappa\lambda\, f(\kappa\mu\lambda, \lambda\mu\kappa)$

$= \mu\kappa\lambda\, f(2,4) = \mu\kappa\lambda(\kappa) = 2$. Hence $f(5,4) \in \{\kappa, \mu\}$. Hence

$f(5,4) = \mu$

A \ B	1 κλμ	2 κμλ	3 λκμ	4 λμκ	5 μκλ	6 μλκ
1 κλμ	κ	κ	κ	κ	κ	κ
2 κμλ	κ	κ	κ	κ	κ	κ
3 λκμ			λ	λ	λ	λ
4 λμκ			λ	λ	λ	λ
5 μκλ				μ	μ	μ
6 μλκ					μ	μ

By monotonicity

A \ B	1 κλμ	2 κμλ	3 λκμ	4 λμκ	5 μκλ	6 μλκ
1 κλμ	κ	κ	κ	κ	κ	κ
2 κμλ	κ	κ	κ	κ	κ	κ
3 λκμ		?	λ	λ	λ	λ
4 λμκ			λ	λ	λ	λ
5 μκλ				μ	μ	μ
6 μλκ				μ	μ	μ

By extremism $f(3,2) \in \{\lambda, \kappa\}$. By IC

$3 = \mu\kappa\lambda(\lambda) = \mu\kappa\lambda\, f(3,5) = \mu\kappa\lambda\, f(\lambda\kappa\mu, \mu\kappa\lambda) \geq$

$\mu\kappa\lambda\, f(\lambda\kappa\mu, \kappa\mu\lambda) = \mu\kappa\lambda\, f(3,2)$. Hence $f(3,2) = \lambda$

Then by monotonicity $f(3,1) = \lambda$, and again by monotonicity

$f(4,1) = f(4,2) = \lambda$.

A \ B	1 κλμ	2 κμλ	3 λκμ	4 λμκ	5 μκλ	6 μλκ
1 κλμ	κ	κ	κ	κ	κ	κ
2 κμλ	κ	κ	κ	κ	κ	κ
3 λκμ	λ	λ	λ	λ	λ	λ
4 λμκ	λ	λ	λ	λ	λ	λ
5 μκλ				μ	μ	μ
6 μλκ			?	μ	μ	μ

By extremism, $f(6,3) \in \{\lambda, \mu\}$. By IC

$2 = \lambda\mu\kappa(\mu) = \lambda\mu\kappa\, f(6,4) = \lambda\mu\kappa\, f(\mu\lambda\kappa, \lambda\mu\kappa) \geq$

$\lambda\mu\kappa\, f(\mu\lambda\kappa, \lambda\kappa\mu) = \lambda\mu\kappa\, f(6,3)$. Hence $f(6,3) \in \{\kappa, \mu\}$.

Hence $f(6,3) = \mu$. By monotonicity, $f(5,3) = \mu$. By monotonicity again, $f(5,2) = f(6,2) = \mu$

A \ B	1 $\kappa\lambda\mu$	2 $\kappa\mu\lambda$	3 $\lambda\kappa\mu$	4 $\lambda\mu\kappa$	5 $\mu\kappa\lambda$	6 $\mu\lambda\kappa$
1 $\kappa\lambda\mu$	κ	κ	κ	κ	κ	κ
2 $\kappa\mu\lambda$	κ	κ	κ	κ	κ	κ
3 $\lambda\kappa\mu$	λ	λ	λ	λ	λ	λ
4 $\lambda\mu\kappa$	λ	λ	λ	λ	λ	λ
5 $\mu\kappa\lambda$	?	μ	μ	μ	μ	μ
6 $\mu\lambda\kappa$		μ	μ	μ	μ	μ

By extremism $f(5,1) \in \{\kappa, \mu\}$. By IC

$$\mu\kappa\lambda\, f(5,1) = \mu\kappa\lambda\, f(\mu\kappa\lambda, \kappa\lambda\mu) \geq \mu\kappa\lambda\, f(\mu\kappa\lambda, \kappa\mu\lambda) =$$

$$= \mu\kappa\lambda\, f(5,2) = \mu\kappa\lambda(\mu) = 1$$

Hence $f(5,1) = \mu$.

By monotonicity $f(6,1) = \mu$.

A \ B	1 $\kappa\lambda\mu$	2 $\kappa\mu\lambda$	3 $\lambda\kappa\mu$	4 $\lambda\mu\kappa$	5 $\mu\kappa\lambda$	6 $\mu\lambda\kappa$
1 $\kappa\lambda\mu$	κ	κ	κ	κ	κ	κ
2 $\kappa\mu\lambda$	κ	κ	κ	κ	κ	κ
3 $\lambda\kappa\mu$	λ	λ	λ	λ	λ	λ
4 $\lambda\mu\kappa$	λ	λ	λ	λ	λ	λ
5 $\mu\kappa\lambda$	μ	μ	μ	μ	μ	μ
6 $\mu\lambda\kappa$	μ	μ	μ	μ	μ	μ

Player A is a dictator