PROBLEM 1

THE ECONOMY

- Two goods,1 and 2, written in this order.
- Two consumers, A and B.

Consumer A

- Consumption set $X_A = \{(A_1, A_2) : A_1 + A_2 \geq \alpha, A_1 \geq 0, A_2 \geq 0\}$
- Endowment vector $\omega_A = [\alpha, 0], \alpha > 0$
- Utility function $u_A = A_1 A_2$

Consumer B

- Consumption set $X_B = \{(B_1, B_2) : B_1 \geq 0, B_2 \geq 0\}$
- Endowment vector $\omega_B = [0, \beta], \beta > 0$
- Utility function $u_B = B_1 B_2$

Parameters α, β

QUESTIONS

Answer the following questions for all allowed values of the parameters α, β

- Compute all competitive equilibria.
- For which values of the parameters, if any, do competitive equilibria exist?

SOLUTION OF PROBLEM 1

1. NAME THE PRICE OF EACH GOOD

$p_1 =$ price of good 1, $p_2 =$ price of good 2 .

Normalize by setting $p_1 = 1$

2. DEFINE CONSUMER INCOMES

$$m_A = \alpha \quad (1)$$

$$m_B = \beta p_2 \quad (2)$$

3. SOLVE THE OPTIMIZATION PROBLEM OF CONSUMER A

$\max u_A = A_1 A_2$
subject to
$A_1 + p_2 A_2 \leq m_A$
$A_1 + A_2 \geq \alpha$
$A_1 \geq 0, A_2 \geq 0$

The solution is

$$[A_1, A_2] = \begin{cases} [\alpha/2, \alpha/2p_2] & \text{IF} \quad p_2 \leq 1 \\ [\alpha, 0] & \text{IF} \quad p_2 \geq 1 \end{cases} \tag{3}$$

4. SOLVE THE OPTIMIZATION PROBLEM OF CONSUMER B

$\max u_B = B_1 B_2$
subject to
$B_1 + p_2 B_2 \leq m_B$
$B_1 \geq 0, B_2 \geq 0$

The solution is

$$[B_1, B_2] = [\beta p_2/2, \beta/2] \tag{4}$$

5. SOLVE THE EQUILIBRIUM CONDITIONS

$$A_1 + B_1 = \alpha, A_2 + B_2 = \beta \tag{5}$$

Competitive equilibrium

if $\alpha > \beta$, then no competitive equilibrium exists

if $\alpha \leq \beta$, then the unique competitive equilibrium is given by

$p_2 = \alpha/\beta$

$[A_1, A_2] = [B_1, B_2] = [\alpha/2, \beta/2]$

(6)

PROBLEM 2

THE ECONOMY

- Two goods, A and X ,written in this order.
- One consumer
- One firm.

The Consumer

- Consumption set $\{(A, X): A \geq 0, X \geq 0\}$
- Endowment vector $\omega = [\bar{A}, \bar{X}]$
- Profit share $\theta = 1$
- Utility function $u = AX$

The firm produces good A out of good X with technology described by the production function

$$\hat{A} = \begin{cases} \hat{X}^2 & \text{if} \quad 0 \leq \hat{X} \leq F \\ F^2 & \text{if} \quad \hat{X} \geq F \end{cases}$$

Parameters: $\bar{A}, \bar{X}, F$.

Conditions on parameters: $0 < F < \bar{X}, \bar{A} \geq 0$

QUESTIONS

Answer the following questions for all allowed parameter values.

- Compute all competitive equilibrium allocations E
- Compute all efficient allocations P
- For which parameter values, if any, is it true that every competitive equilibrium allocation is efficient, i.e., that $E \subset P$?
- Compute the set of decentralizable efficient allocations $P \bigcap E$
- For which parameter values, if any, is it true that all efficient allocations are decentralizable, i.e., that $P \subseteq E$?

SOLUTION OF PROBLEM 3

COMPETITIVE EQUILIBRIA

1.NAME THE PRICE OF EACH GOOD

$p =$ price of commodity A,w=price of commodity X.

2. DEFINE CONSUMER INCOME

$$M = p\bar{A} + w\bar{X} + \Pi \tag{7}$$

3. SOLVE THE OPTIMIZATION PROBLEM OF THE CONSUMER

max $u = AX$,subject to $pA + wX \leq M$, $A \geq 0, X \geq 0$

Variables: A, X

parameters: w, M, p

Conditions on parameters: $w > 0, M > 0, p > 0$

The solution is

$$\left(A, X\right) = \left(\frac{M}{2p}, \frac{M}{2w}\right) \tag{8}$$

4. SOLVE THE OPTIMIZATION PROBLEM OF THE FIRM

$$\max \Pi = p\hat{A} - w\hat{X} = \begin{cases} pX^2 - wX & \text{if} \quad X \leq F \\ pF^2 - wX & \text{if} \quad X \geq F \end{cases}$$

subject to $\hat{X} \geq 0, \hat{A} \geq 0$

Variables:$\hat{X}, \hat{A}$

parameters: p, w, F

conditions on parameters:$w > 0, F \geq 0, p > 0$

The solution is

$$\left(\hat{A}, \hat{X}, \Pi\right) = \begin{cases} (F^2, F, pF^2 - wF) & \text{if} \quad \frac{w}{p} < F \\ \{(0,0,0), (F^2, F, 0)\} & \text{if} \quad \frac{w}{p} = F \\ (0,0,0) & \text{if} \quad \frac{w}{p} > F \end{cases} \tag{9}$$

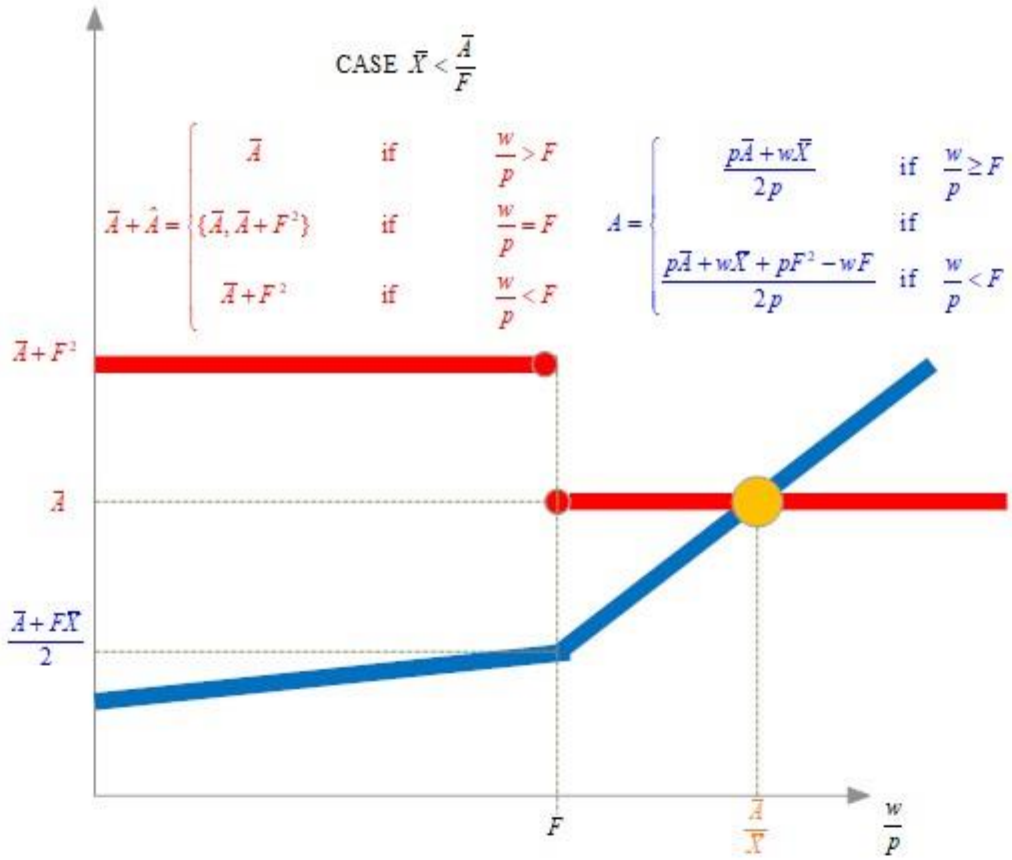
5. SOLVE THE EQUILIBRIUM CONDITIONS

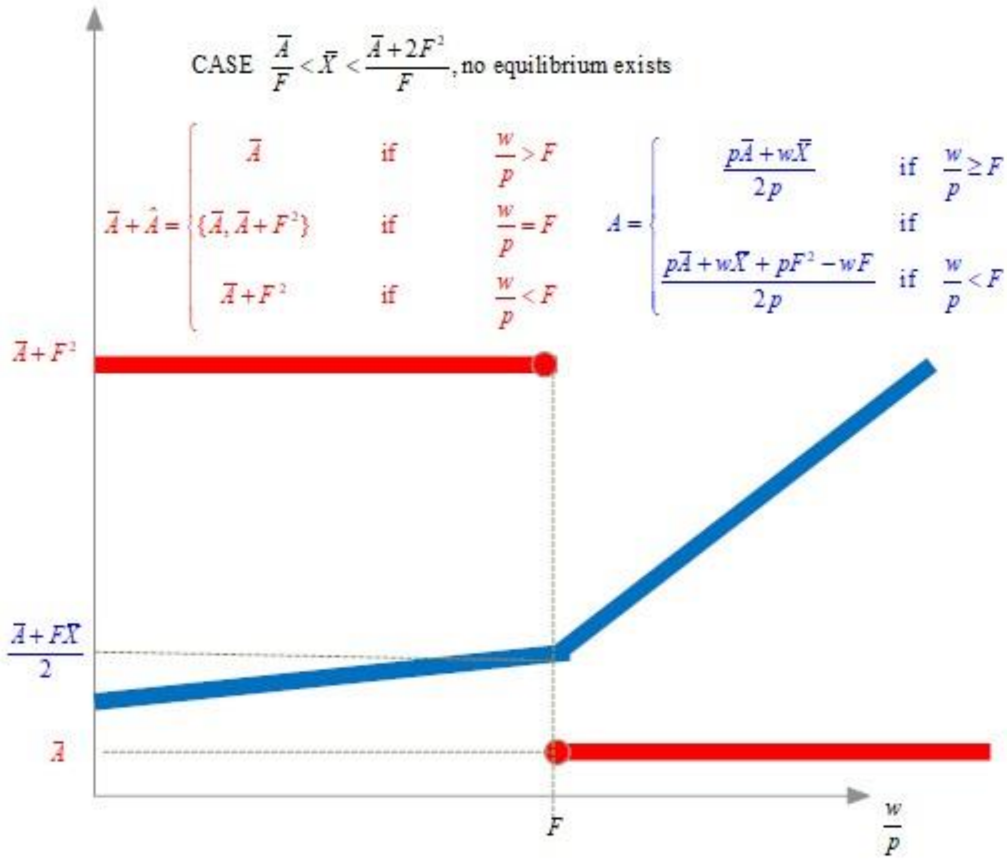
$$\boxed{\begin{array}{ccc} \text{demand} & = & \text{supply} \\ A & = & \bar{A}+\hat{A} \\ \hat{X}+X & = & \bar{X} \end{array}} \qquad (10)$$

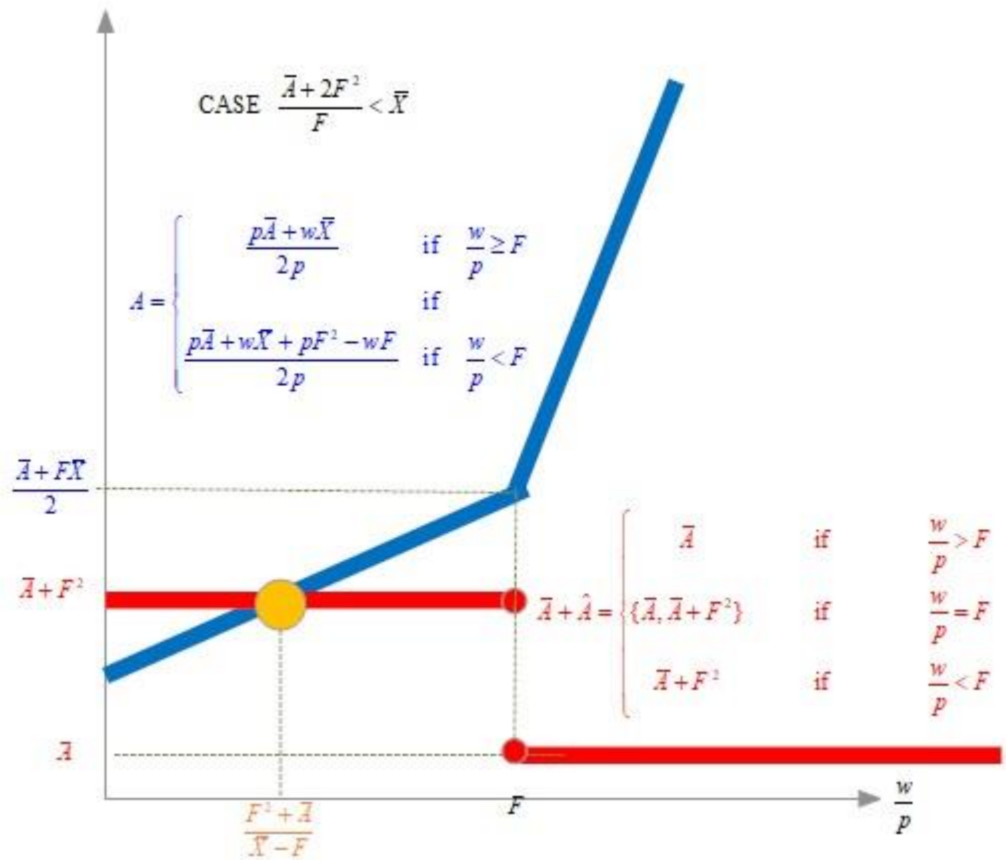
By (10),(9),(8),(7) we obtain

$$\boxed{\begin{array}{l} \boxed{\text{equilibrium prices}} \\ \dfrac{w}{p}=\begin{cases} \dfrac{\bar{A}}{\bar{X}} & \text{if} \quad \bar{X}\le\dfrac{\bar{A}}{F} \\ \text{no equilibrium exists} & \text{if} \quad \dfrac{\bar{A}}{F}<\bar{X}<\dfrac{\bar{A}+2F^2}{F} \\ \dfrac{F^2+\bar{A}}{\bar{X}-F} & \text{if} \quad \bar{X}\ge\dfrac{\bar{A}+2F^2}{F} \end{cases} \end{array}} \qquad (11)$$

$$\boxed{\begin{array}{l} \text{Equilibrium allocations } E=\{[A,X,\hat{A},\hat{X}]\} \\ E=\begin{cases} \{[\bar{A},\bar{X},0,0]\} & \text{if} \quad \bar{X}\le\dfrac{\bar{A}}{F} \\ \varnothing & \text{if} \quad \dfrac{\bar{A}}{F}<\bar{X}<\dfrac{\bar{A}+2F^2}{F} \\ \{[\bar{A}+F^2,\bar{X}-F,F^2,F]\} & \text{if} \quad \bar{X}\ge\dfrac{\bar{A}+2F^2}{F} \end{cases} \end{array}} \qquad (12)$$


CASE $\bar{X} < \frac{\bar{A}}{F}$
$\bar{A} + \hat{A} = \begin{cases} \bar{A} & \text{if } \frac{w}{p} > F \\ \{\bar{A}, \bar{A}+F^2\} & \text{if } \frac{w}{p} = F \\ \bar{A}+F^2 & \text{if } \frac{w}{p} < F \end{cases}$
$A = \begin{cases} \frac{p\bar{A}+w\bar{X}}{2p} & \text{if } \frac{w}{p} \geq F \\ & \text{if} \\ \frac{p\bar{A}+w\bar{X}+pF^2-wF}{2p} & \text{if } \frac{w}{p} < F \end{cases}$
$\bar{A}+F^2$
$\bar{A}$
$\frac{\bar{A}+F\bar{X}}{2}$
F
$\frac{\bar{A}}{\bar{X}}$
$\frac{w}{p}$


CASE $\frac{\bar{A}}{F} < \bar{X} < \frac{\bar{A}+2F^2}{F}$, no equilibrium exists
$\bar{A}+\hat{A} = \begin{cases} \bar{A} & \text{if } \frac{w}{p} > F \\ \{\bar{A}, \bar{A}+F^2\} & \text{if } \frac{w}{p} = F \\ \bar{A}+F^2 & \text{if } \frac{w}{p} < F \end{cases}$
$A = \begin{cases} \frac{p\bar{A}+w\bar{X}}{2p} & \text{if } \frac{w}{p} \geq F \\ & \text{if} \\ \frac{p\bar{A}+w\bar{X}+pF^2-wF}{2p} & \text{if } \frac{w}{p} < F \end{cases}$
$\bar{A}+F^2$
$\frac{\bar{A}+F\bar{X}}{2}$
$\bar{A}$
F
$\frac{w}{p}$



EFFICIENT POINTS

Efficient points will be the global maxima of the following maximization problem

$$
\begin{aligned}
&\max u = AX \\
&\text{subject} \quad \text{to} \\
&\quad A \quad \leq \quad \bar{A} + \hat{A} \\
&\hat{X} + X \quad \leq \quad \bar{X} \\
&\hat{A} = \begin{cases} \hat{X}^2 & \text{if} \quad 0 \leq \hat{X} \leq F \\ F^2 & \text{if} \quad \hat{X} \geq F \end{cases} \\
&A \geq 0, X \geq 0, \hat{A} \geq 0, \hat{X} \geq 0 \qquad (13) \\
&\text{variables} \quad A, X, \hat{A}, \hat{X} \\
&\text{parameters} \quad F, \bar{A}, \bar{X} \\
&\text{conditions on parameters} \quad 0 < F < \bar{X}, \bar{A} \geq 0
\end{aligned}
$$

At any global maximum of (13), $\hat{X} \leq F, \hat{A} = \hat{X}^2$,hence (13) is equivalent to

$$
\begin{aligned}
&\max u = AX \\
&\text{subject} \quad \text{to} \\
&\quad A \quad \leq \quad \bar{A} + \hat{X}^2 \\
&\hat{X} + X \quad \leq \quad \bar{X} \\
&\hat{X} \leq F \\
&A \geq 0, X \geq 0, \hat{X} \geq 0 \qquad (14) \\
&\text{variables} \quad A, X, \hat{X} \\
&\text{parameters} \quad F, \bar{A}, \bar{X} \\
&\text{conditions on parameters} \quad 0 < F < \bar{X}, \bar{A} \geq 0
\end{aligned}
$$

Note that (14) has always global maxima, and therefore the set of efficient allocations is nonempty for all parameter values.

At any global maximum of (14) the following conditions are satisfied

Every efficient allocation satisfies
$\hat{A} = \hat{X}^2, \hat{X} \leq F$
$A = \bar{A} + \hat{X}^2, \hat{X} + X = \bar{X}$

(15)

Note that, by (15),the set of efficient allocations is uniquely determined by the value of $\hat{X}$.

By (15),(14) efficient points will be uniquely determined by the global maxima of the following maximization problem

$$
\begin{aligned}
&\max \text{ u} = \left(\bar{A} + \hat{X}^2\right)\left(\bar{X} - \hat{X}\right) \\
&\text{subject to } \; 0 \le \hat{X} \le F \\
&\text{variables } \; \hat{X} \\
&\text{parameters } \; F, \bar{A}, \bar{X} \\
&\text{conditions on parameters } \; 0 < F < \bar{X}, \bar{A} \ge 0
\end{aligned}
\tag{16}
$$

Removing the constant term from the objective function in (16), efficient points will be uniquely determined by the global maxima of the following maximization problem

$$
\begin{aligned}
&\max \hat{X}g\left(\hat{X}\right) = \hat{X}\left(\bar{X}\hat{X} - \bar{A} - \hat{X}^2\right) \\
&\text{subject to } \; 0 \le \hat{X} \le F \\
&\text{variables } \; \hat{X} \\
&\text{parameters } \; F, \bar{A}, \bar{X} \\
&\text{conditions on parameters } \; 0 < F < \bar{X}, \bar{A} \ge 0
\end{aligned}
\tag{17}
$$

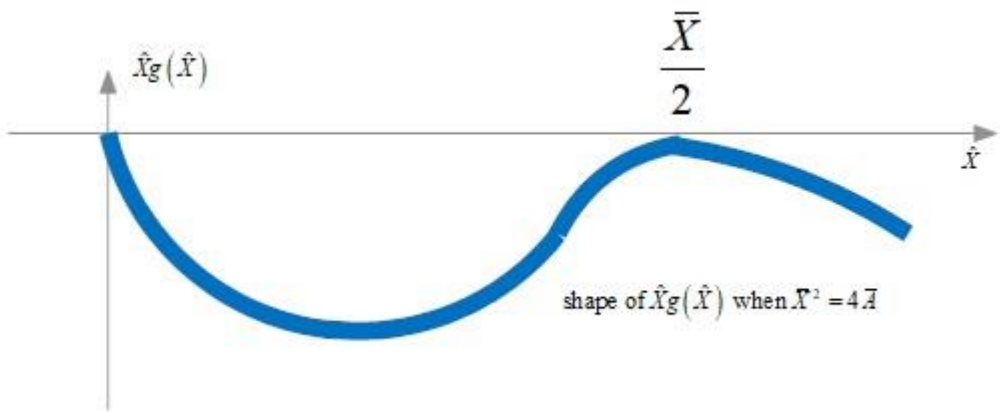
properties of g

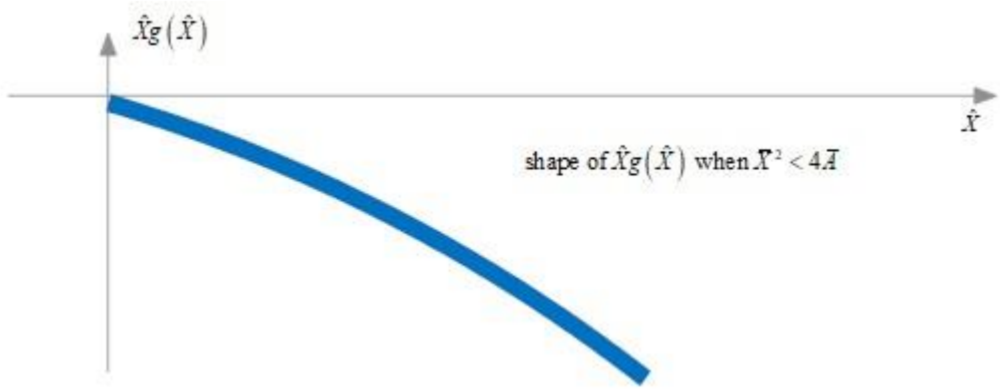
$$
\begin{aligned}
&\bar{X}^2 < 4\bar{A} \Rightarrow g \text{ is always negative} \\
&\bar{X}^2 = 4\bar{A} \Rightarrow g \text{ is always negative except that } g\left(\frac{\bar{X}}{2}\right) = 0 \\
&\bar{X}^2 > 4\bar{A} \Rightarrow \\
&\text{roots of g } \; r_1 = \frac{\bar{X}}{2} - \frac{\sqrt{\bar{X}^2 - 4\bar{A}}}{2}, r_2 = \frac{\bar{X}}{2} + \frac{\sqrt{\bar{X}^2 - 4\bar{A}}}{2} \\
&\text{Global maximum of g } \; \hat{X} = \frac{\bar{X}}{2} \\
&\text{maximum value of g } = \frac{\bar{X}^2}{4} - \bar{A} > 0
\end{aligned}
\tag{18}
$$

Hence

efficient values of $\hat{X}$

$$\hat{X} = \begin{cases} 0 & \text{if } \bar{X}^2 < 4\bar{A}, \text{OR} \left(\bar{X}^2 = 4\bar{A}, \bar{X} > 2F\right) \\ \left\{0, \frac{\bar{X}}{2}\right\} & \text{if } \bar{X}^2 = 4\bar{A}, \bar{X} \le 2F \\ ? & \text{if } \bar{X}^2 > 4\bar{A} \end{cases} \quad (19)$$

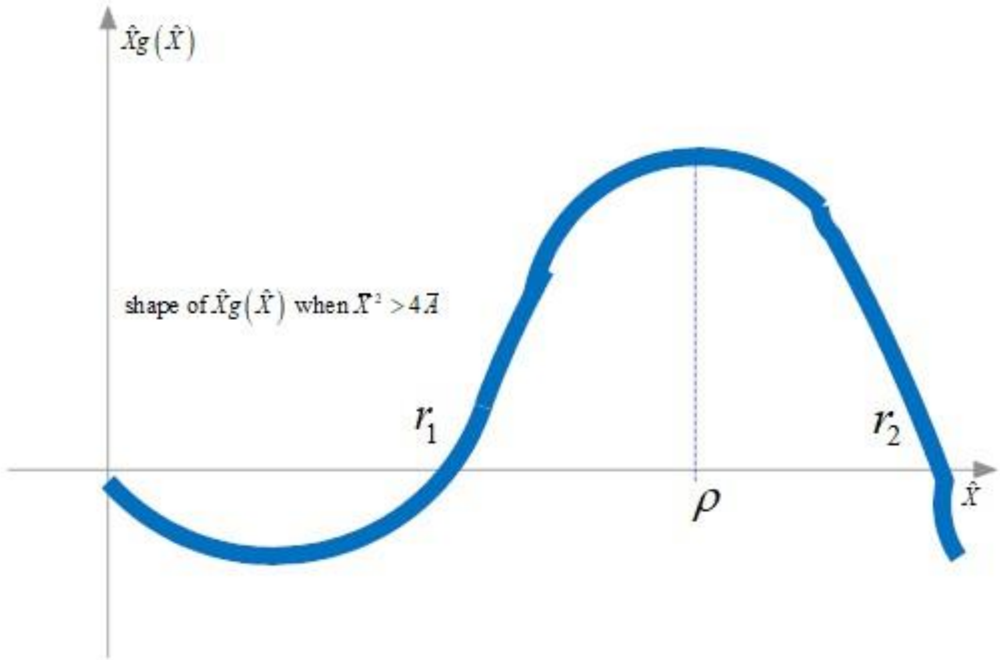




properties of $\hat{X}g(\hat{X})$ when $\bar{X}^2 > 4\bar{A}$

roots $0, r_1, r_2$

global maximum $\hat{X} = \rho \triangleq \frac{\bar{X}}{3} + \frac{\sqrt{\bar{X}^2 - 3\bar{A}}}{3}$ (20)



efficient values of $\hat{X}$ when $\bar{X}^2 > 4\bar{A}$

$$\hat{X} = \begin{cases} 0 & \text{if} \quad F < r_1 \\ \{0, F\} & \text{if} \quad F = r_1 \\ F & \text{if} \quad r_1 < F \le \rho \\ \rho & \text{if} \quad F > \rho \end{cases} \tag{21}$$

By (21),(19),(18) we conclude

efficient values of $\hat{X}$

$$\hat{X} = \begin{cases} 0 & \text{if} \quad \bar{X}^2 < 4\bar{A}, \text{or } \left(\bar{X}^2 = 4\bar{A}, \bar{X} > 2F\right), \text{or } \left(\bar{X}^2 > 4\bar{A}, F < r_1\right) \\ \left\{0, \frac{\bar{X}}{2}\right\} & \text{if} \quad \bar{X}^2 = 4\bar{A}, \bar{X} \le 2F \\ \{0, F\} & \text{if} \quad \bar{X}^2 > 4\bar{A}, F = r_1 \\ F & \text{if} \quad \bar{X}^2 > 4\bar{A}, r_1 < F \le \rho \\ \rho & \text{if} \quad \bar{X}^2 > 4\bar{A}, F > \rho \end{cases} \tag{22}$$

By (22),(15),the set of efficient allocations is given by

Efficient allocations $P=\{[A,X,\hat{A},\hat{X}]\}$

$$P=\begin{cases}\left\{[\bar{A},\bar{X},0,0]\right\} & \text{if} \quad \bar{X}^2<4\bar{A} \text{ OR } \left(\bar{X}^2=4\bar{A},\bar{X}>2F\right) \text{ OR } \left(\bar{X}^2>4\bar{A},F<r_1\right)\\ \left\{[\bar{A},\bar{X},0,0],[\frac{\bar{X}^2}{2},\frac{\bar{X}}{2},\left(\frac{\bar{X}}{2}\right)^2,\frac{\bar{X}}{2}]\right\} & \text{if} \quad \bar{X}^2=4\bar{A},\bar{X}<2F\\ \left\{[\bar{A},\bar{X},0,0],[\bar{A}+F^2,\bar{X}-F,F^2,F]\right\} & \text{if} \quad \left(\bar{X}^2=4\bar{A},\bar{X}=2F\right) \text{ OR } \left(\bar{X}^2>4\bar{A},F=r_1\right)\\ \left\{[\bar{A}+F^2,\bar{X}-F,F^2,F]\right\} & \text{if} \quad \bar{X}^2>4\bar{A},r_1<F\le\rho\\ \left\{[\bar{A}+\rho^2,\bar{X}-\rho,\rho^2,\rho]\right\} & \text{if} \quad \bar{X}^2>4\bar{A},F>\rho\end{cases}$$

(23)

COMPETITIVE ALLOCATIONS ARE EFFICIENT.

By (12) $\hat{X}=0$ in equilibrium implies $\bar{X}\le\frac{\bar{A}}{F}$. $\bar{X}\le\frac{\bar{A}}{F}$ is incompatible with $\bar{X}^2=4\bar{A},\bar{X}<2F$;and $\bar{X}\le\frac{\bar{A}}{F}$ is incompatible with $\bar{X}^2>4\bar{A},r_1<F$ .In all other cases $\hat{X}=0$ is efficient.

By (12) $\hat{X}=F$ in equilibrium implies $\bar{X}\ge\frac{\bar{A}+2F^2}{F}$. $\bar{X}\ge\frac{\bar{A}+2F^2}{F}$ is incompatible with $\bar{X}^2<4\bar{A}$ OR $\left(\bar{X}^2=4\bar{A},\bar{X}>2F\right)$ OR $\left(\bar{X}^2>4\bar{A},F<r_1\right)$; $\bar{X}\ge\frac{\bar{A}+2F^2}{F}$ is incompatible with $\bar{X}^2=4\bar{A},\bar{X}<2F$; $\bar{X}\ge\frac{\bar{A}+2F^2}{F}$ is incompatible with $\bar{X}^2>4\bar{A},F>\rho$ .In all other cases $\hat{X}=F$ is efficient. We conclude that in all cases,

$$E\subseteq P, P\cap E=E \qquad (24)$$

EFFICIENT POINTS ARE NOT ALWAYS DECENTRALIZABLE.

In the case $\frac{\bar{A}}{F}<\bar{X}<\frac{\bar{A}+2F^2}{F}$,we have $P\neq\varnothing,E=\varnothing$ hence $P\not\subset E$ .In the case $\bar{X}^2>4\bar{A},r_1<F\le\rho,\bar{X}\ge\frac{\bar{A}+2F^2}{F}$ we have $P=E=\left\{[\bar{A}+F^2,\bar{X}-F,F^2,F]\right\}$.

PROBLEM 3

THE ECONOMY

- Two goods, A and B,written in this order.
- Two consumers,1 and 2

Consumer 1

- Consumption set $X_1 = \{(A_1, B_1) : A_1 \geq 0, B_1 \geq 0\}$
- Endowment vector $\omega_1 = [5,1]$
- Utility function $u_1(A_1, B_1, A_2, B_2) = A_1 (B_1)^2$

Consumer 2

- Consumption set $X_2 = \{(A_2, B_2) : A_2 \geq 0, B_2 \geq 0\}$
- Endowment vector $\omega_2 = [0,2]$
- Utility function $u_2(A_1, B_1, A_2, B_2) = 4\log A_2 + 2\log B_2 - A_1$

QUESTIONS

1.Compute all efficient points

2.Compute all competitive equilibria under the following type of taxation: buyers of good A pay a tax t per unit of quantity purchased. The tax rate t can be positive, zero, or negative. Tax revenue is split equally between consumers with a lump-sum transfer.

3.Draw the equilibrium utility level of each consumer as a function of the tax rate t

4. Draw the equilibrium level of tax revenue as a function of the tax rate t

5.For which levels of the tax rate t is competitive equilibrium efficient?

SOLUTION OF PROBLEM 3

EFFICIENT ALLOCATIONS

We solve one of the auxiliary problems defined by any vector optimization problem.

auxiliary maximization problem for consumer 2

objective function

$u_2 = 4\log A_2 + 2\log B_2 - A_1$

The feasible set is defined by the constraints

$\log A_1 + 2\log B_1 \geq \theta_1$ (25)

$A_1 + A_2 \leq 5, B_1 + B_2 \leq 3$

$A_1 \geq 0, A_2 \geq 0, B_1 \geq 0, B_2 \geq 0$

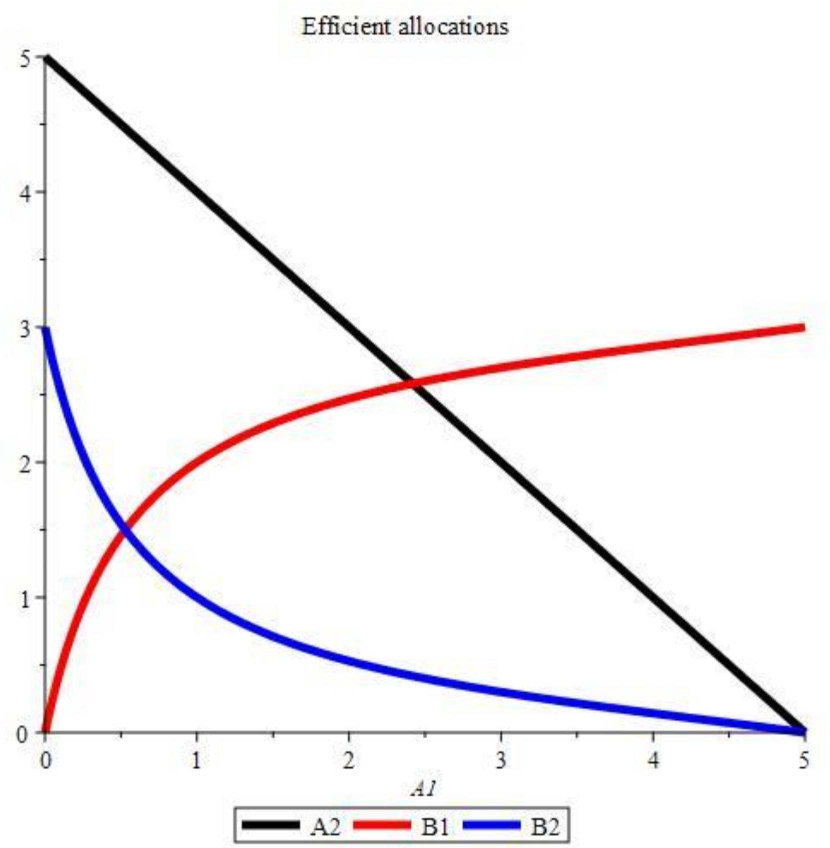
variables A_1, A_2, B_1, B_2

parameters θ_1

The global maxima of this problem are described below. Since they are essentially unique, they also define the set of efficient allocations.

Efficient allocations

$$A_2 = 5 - A_1, B_1 = \frac{3A_1(A_1 - 9)}{A_1^2 - 8A_1 - 5}, B_2 = \frac{3(A_1 - 5)}{A_1^2 - 8A_1 - 5}, 0 \leq A_1 \leq 5 \quad (26)$$



COMPETITIVE EQUILIBRIUM

1. NAME THE PRICE OF EACH GOOD

$p_A =$ price of good A, $p_B =$ price of good B.

Normalize by setting $p_A = 1$ .Rename p_B into p

2. DEFINE CONSUMER INCOMES

$$m_1 = 5 + p + T \quad (27)$$

$$m_2 = 2p + T \quad (28)$$

3. SOLVE THE OPTIMIZATION PROBLEM OF CONSUMER 1

max $u_1 = \log A_1 + 2\log B_1$
subject to
$A_1 + pB_1 \le m_1$
$A_1 \ge 0, B_1 \ge 0$
variables A_1, B_1
parameters p, m_1
conditions on parameters $p > 0, m_1 > 0$

The solution is

$$\left[A_1, B_1\right] = [m_1/3, 2m_1/3p] \quad (29)$$

4. SOLVE THE OPTIMIZATION PROBLEM OF CONSUMER 2

max $u_2 = 4\log A_2 + 2\log B_2 - A_1$
subject to
$(1+t)A_2 + pB_2 \le m_2$
$B_1 \ge 0, B_2 \ge 0$
variables A_2, B_2
parameters p, m_2, t
conditions on parameters $p > 0, m_1 > 0, t > -1$

The solution is

$$[A_2, B_2] = [2m_2 / 3(1+t), m_2 / 3p] \tag{30}$$

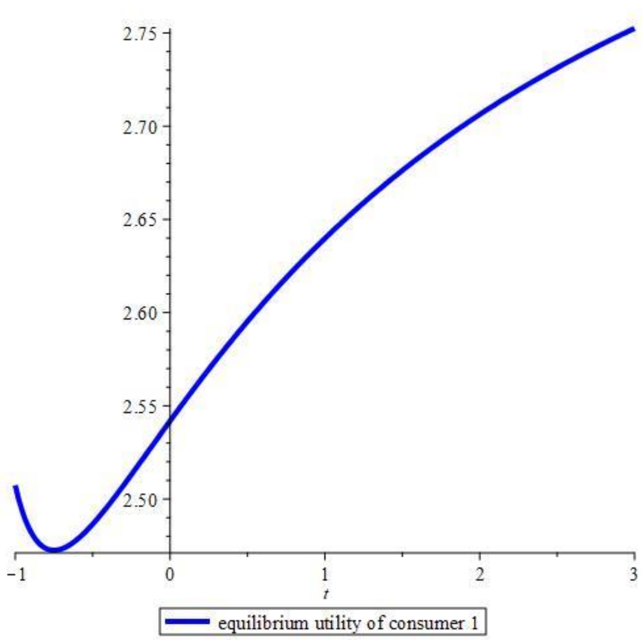
5. SOLVE THE EQUILIBRIUM CONDITIONS

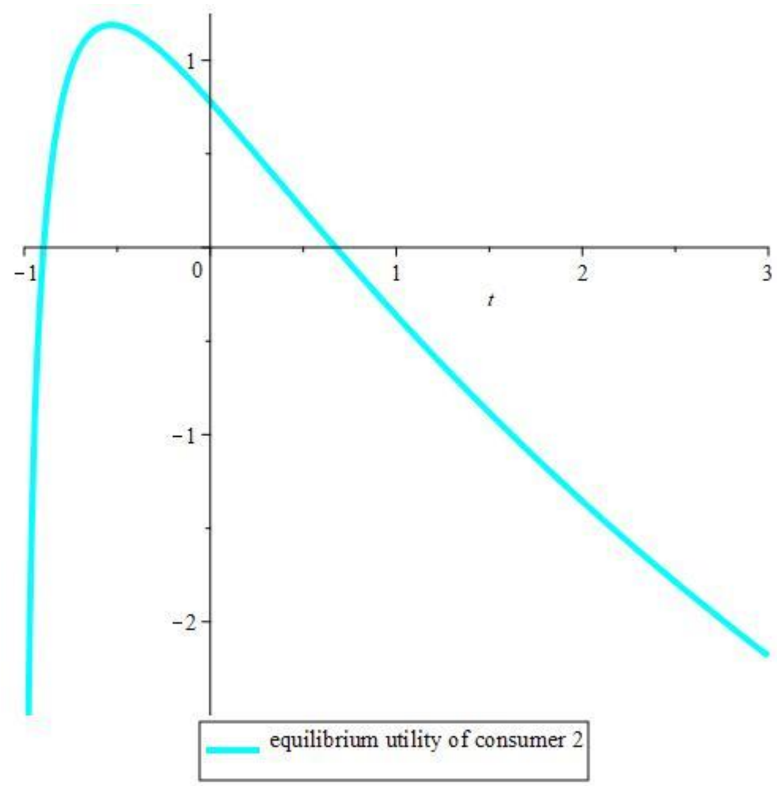
$$A_1 + A_2 = 5, B_1 + B_2 = 3 \tag{31}$$

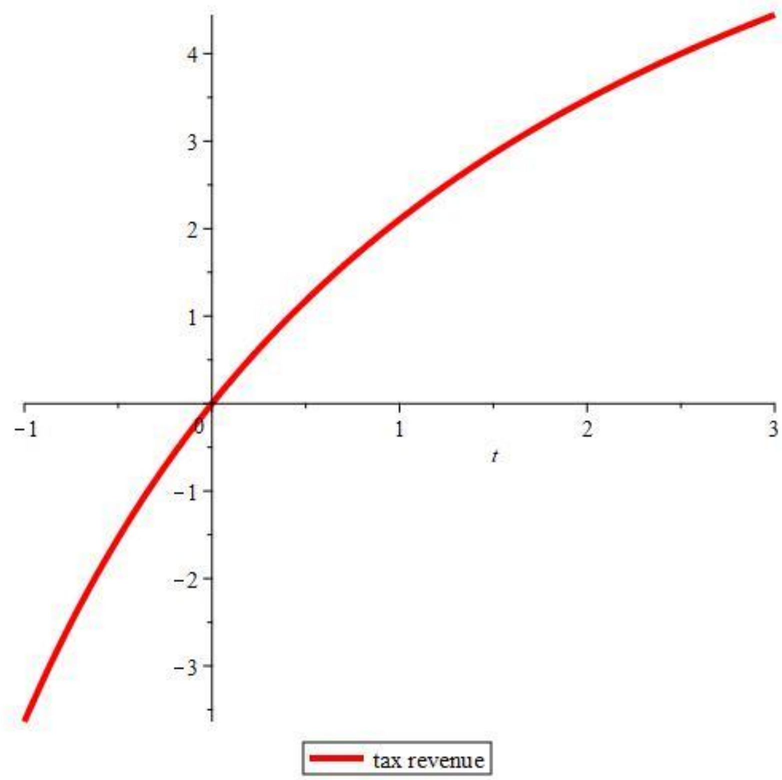
Competitive equilibrium

$$\begin{aligned} &p = \frac{10(2t+3)}{4t+15}, T = \frac{20t}{4t+15} \\ &A_1 = \frac{5(7+4t)}{4t+15}, A_2 = \frac{40}{4t+15}, B_1 = \frac{7+4t}{2t+3}, B_2 = \frac{2(1+t)}{2t+3} \\ &u_1 = \ln(5(7+4t)/(4t+15)) + 2\ln((7+4t)/(2t+3)) \\ &u_2 = 4\ln\left(\frac{40}{4t+15}\right) + 2\ln\left(\frac{2(1+t)}{2t+3}\right) - \frac{5(7+4t)}{4t+15} \end{aligned} \tag{32}$$

EQUILIBRIUM LEVELS OF UTILITY AND TAX REVENUE AS FUNCTIONS OF THE TAX RATE




1
0
−1
−2
−1
1
2
3
t
equilibrium utility of consumer 2


4
3
2
1
−1
−2
−3
−1
1
2
3
t
tax revenue

OPTIMAL LEVEL OF THE TAX RATE

We substitute (32) into (26).We obtain the equations

$$\boxed{\begin{array}{l}\text{equilibrium allocation is efficient when}\\ \dfrac{2+2t}{2t+3}=\dfrac{24t+90}{16t^2+136t+205}\\ \\ \dfrac{7+4t}{2t+3}=\dfrac{48t^2+384t+525}{16t^2+136t+205}\end{array}} \qquad (33)$$

Solving (33) we obtain

$$\boxed{\begin{array}{l}\text{OPTIMAL TAX POLICY}\\ t_{optimal}=-\dfrac{25}{8}+\dfrac{\sqrt{465}}{8}\simeq -0.429517669\\ T_{optimal}=\dfrac{-125+5\sqrt{465}}{5+\sqrt{465}}\approx -0.6467699978\end{array}} \qquad (34)$$