THE ECONOMY

- Two goods, A and x, written in this order.
- Two consumers, 1 and 2.
- One firm

Consumer 1

- Consumption set $X_1 = \{(A_1, x_1) \in \mathbb{R}^2 : A_1 \geq 0, x_1 \geq 0\}$
- Endowment vector $\omega_1 = [0, 4]$, Profit share 1/2
- Utility function $u_1 = \frac{1}{2}\log x_1 + \frac{1}{2}\log A_1 + \theta \log(A_2)$

Consumer 2

- Consumption set $X_2 = \{(A_2, x_2) \in \mathbb{R}^2 : A_2 \geq 0, x_2 \geq 0\}$
- Endowment vector $\omega_2 = [0, 4]$, Profit share 1/2
- Utility function $u_2 = \frac{1}{2}\log x_2 + \frac{1}{2}\log A_2 + \theta \log(A_1)$

The firm produces good A out of good x with technology described by the production function $A = \frac{x}{2}$

Policy measures

The firm receives a subsidy s per unit of output. Each consumer pays a lump-sum tax T

Parameters θ and s .For consumers and the firm, T is also a parameter. Its value will be determined in equilibrium.

Conditions on parameters $\theta > 0, T \geq 0, s \geq 0$

QUESTIONS

Answer the following questions for all allowed values of the parameters.

- Compute all competitive equilibria as functions of the parameters (θ, s)
- Compute all efficient allocations.
- For which values of the parameters (θ, s) , if any, are competitive equilibria efficient?

SOLUTION

EFFICIENT ALLOCATIONS

The set of feasible allocations

$$FA = \left\{ \left(A_1, X_1, A_2, X_2, A \right) \in R_+^5 : A_1 + A_2 \le A, X_1 + X_2 + 2A \le 8 \right\} \qquad (1)$$

is convex, and the objective functions u_1, u_2 are concave. Hence the utility possibility set is convex, the method of the linear SWF is **complete**, and we can try to compute efficient points by solving, for all values of the parameter $\beta \in [0,1]$,the following max problem

$$\begin{aligned} &\max W_\beta = \beta u_1 + \left(1-\beta\right) u_2 \\ &\text{subject to } \left(A_1, X_1, A_2, X_2, A \right) \in FA \end{aligned} \qquad (2)$$

$$\boxed{\begin{aligned} &\text{global maxima } G\left(\beta\right) \text{of } \left(W_\beta, FA\right) \\ &A = \frac{4\theta + 2}{\theta + 1} \\ &A_1 = \frac{4\left(1-\beta\right)\theta + 2\beta}{\theta + 1}, X_1 = \frac{4\beta}{\theta + 1} \\ &A_2 = \frac{4\theta\beta - 2\beta + 2}{\theta + 1}, X_2 = \frac{4\left(1-\beta\right)}{\theta + 1} \end{aligned}} \qquad (3)$$

Since $G\left(\beta\right)$ is a singleton for all $\beta \in [0,1]$,the method of the linear SWF is **sound.** Hence the set of efficient points is $\bigcup_{0 \le \beta \le 1} G\left(\beta\right)$

$$\boxed{\begin{aligned} &\text{efficient allocations} \\ &A = \frac{4\theta + 2}{\theta + 1} \\ &A_1 = \frac{4\left(1-\beta\right)\theta + 2\beta}{\theta + 1}, X_1 = \frac{4\beta}{\theta + 1} \\ &A_2 = \frac{4\theta\beta - 2\beta + 2}{\theta + 1}, X_2 = \frac{4\left(1-\beta\right)}{\theta + 1} \\ &0 \le \beta \le 1 \end{aligned}} \qquad (4)$$

EQUILIBRIA

1.NAME THE PRICE OF EACH GOOD

$p =$ price of good A,w=price of good X. Normalize $p = 1$

2. DEFINE CONSUMER INCOMES

$$M_1 = 4w + \Pi / 2 - T, M_2 = 4w + \Pi / 2 - T \tag{5}$$

3. SOLVE THE OPTIMIZATION PROBLEMS OF CONSUMERS

max u_1 subject to $A_1 + wX_1 \le M_1, A_1 \ge 0, X_1 \ge 0$
max u_2 subject to $A_2 + wX_2 \le M_2, A_2 \ge 0, X_2 \ge 0$

The solutions are

$$(A_1, X_1) = \left(\frac{4w + \Pi / 2 - T}{2}, \frac{4w + \Pi / 2 - T}{2w} \right) \tag{6}$$

$$(A_2, X_2) = \left(\frac{4w + \Pi / 2 - T}{2}, \frac{4w + \Pi / 2 - T}{2w} \right) \tag{7}$$

4. SOLVE THE OPTIMIZATION PROBLEM OF THE FIRM

max $\Pi = (1+s)A - wX, X \ge 0, A = X / 2$

The solution is

$$(X, A, \Pi) = \begin{cases} \text{none} & \text{if} \quad w < \frac{1+s}{2} \\ (2A, A, 0), A \ge 0 & \text{if} \quad w = \frac{1+s}{2} \\ (0,0,0) & \text{if} \quad w > \frac{1+s}{2} \end{cases} \tag{8}$$

5. SOLVE THE EQUILIBRIUM CONDITIONS

$$A = A_1 + A_2, 8 = X_1 + X_2 + X \tag{9}$$

$$\boxed{\begin{array}{l}\text{equilibrium with an output subsidy}\\ w=\frac{1}{2}+\frac{s}{2},T=\frac{2s(1+s)}{s+2},\Pi=0\\ \boxed{\begin{array}{l}\text{Equilibrium allocation } E(\theta,s)\\ A=\frac{4(1+s)}{s+2},X=\frac{8(1+s)}{s+2}\\ A_1=\frac{2(1+s)}{s+2},X_1=\frac{4}{s+2}\\ A_2=\frac{2(1+s)}{s+2},X_2=\frac{4}{s+2}\end{array}}\end{array}} \tag{10}$$

CORRECTION OF EQUILIBRIA TO ATTAIN EFFICIENCY.

Step 1. We substitute (10) into (4) and obtain

$$\boxed{\begin{array}{l}\frac{2+2s}{s+2}=\frac{(4-4\beta)\theta+2\beta}{\theta+1}\\ \frac{2+2s}{s+2}=\frac{4\theta\beta-2\beta+2}{\theta+1}\\ \frac{4+4s}{s+2}=\frac{4\theta+2}{\theta+1}\\ \frac{4}{s+2}=\frac{4-4\beta}{\theta+1}\\ \frac{4}{s+2}=\frac{4\beta}{\theta+1}\end{array}} \tag{11}$$

Step 2. we solve (11) with respect to $\{s,\theta\}$

$$\beta=\frac{1}{2},s=2\theta \tag{12}$$

Step 3. we substitute (12) into (10)

optimal tax-subsidy policy

$$s = 2\theta, T = \frac{4\theta^2 + 2\theta}{\theta + 1}$$

equilibrium allocation induced by the optimal tax-subsidy policy

$$A = \frac{4\theta + 2}{\theta + 1}, X = 2\frac{4\theta + 2}{\theta + 1}$$

$$A_1 = \frac{2\theta + 1}{\theta + 1}, X_1 = \frac{2}{\theta + 1}$$

$$A_2 = \frac{2\theta + 1}{\theta + 1}, X_2 = \frac{2}{\theta + 1} \qquad (13)$$