TAKEHOME EXAM

Please answer all questions. Omitting calculations is OK.

PROBLEM 1

THE ECONOMY

Consider a three-period economy with

●two consumers, namely Ψ and Ω

●three goods, namely A, L, B

● Two firms, namely α and β

firm α produces good A out of good L with production function

$$A_\alpha = cL_\alpha, c > 0 \tag{1}$$

where L_α is the quantity of good L used as an input in the first period, and A_α is the quantity of good A produced in the second period, and c is a parameter.

firm β produces good B out of good A with production function

$$B_\beta = 4\sqrt{A_\beta} \tag{2}$$

where A_β is the quantity of good A used as an input in the second period, and B_β is the quantity of good B produced in the third period.

Consumer Ψ

● consumption set: all variables ≥ 0

● utility function

$$u_\Psi = A_2^\Psi B_3^\Psi \tag{3}$$

where A_2^Ψ is the quantity of good A consumed by Ψ in the second period, and B_3^Ψ is the quantity of good B consumed by Ψ in the third period.

●endowment: $\bar{L} > 0$ units of good L in the first period only. No endowment of any other good in any period.

● profit shares: sole owner of firm α ,no share in firm β

consumer Ω

● consumption set: all variables ≥ 0

● utility function

$$u_{\Omega} = L_1^{\Omega} \tag{4}$$

where L_1^{Ω} is the quantity of good L consumed by Ω in the first period.

●endowment: no endowment of any good in any period.

● profit shares: sole owner of firm β ,no share in firm α

Policy

Firm α pays a tax $t_{\alpha} \geq 0$ per unit of input used.

Firm β receives a subsidy $s_{\beta} \geq 0$ per unit of input used.

Firm β pays a tax $\tau, 0 \leq \tau < 1$ per unit of profit earned.

QUESTIONS

Answer the following questions for all allowed parameter values

1.Compute all efficient points.

2.Compute all competitive equilibria.

3.For which parameter values are competitive equilibria efficient?

answers to problem 1

1.EFFICIENT ALLOCATIONS

We scalarize the problem of computation of efficient points, using the general method of minimum guaranteed utility levels, i.e., we solve the following maximization problem for all values of the parameter θ

$$\begin{aligned}
&\max u_\Psi = A_2^\Psi B_3^\Psi \\
&\text{subject to} \\
&L_1^\Omega \ge \theta \\
&L_1^\Omega + L_\alpha \le \bar{L} \\
&A_2^\Psi + A_\beta \le A_\alpha = cL_\alpha \\
&B_3^\Psi \le B_\beta = 4\sqrt{A_\beta} \\
&\text{all variables } \ge 0 \\
&\text{variables } A_2^\Psi, B_3^\Psi, L_1^\Omega, L_\alpha, A_2^\Psi, A_\beta, A_\alpha, B_\beta \\
&\text{parameters } c, \theta \\
&\text{conditions on parameters } c > 0
\end{aligned} \tag{5}$$

$$\boxed{\begin{aligned}
&\underline{\text{efficient allocations}} \\
&0 \le L_1^\Omega \le \bar{L} \\
&A_2^\Psi = \frac{2c\left(\bar{L} - L_1^\Omega\right)}{3}, L_\alpha = \bar{L} - L_1^\Omega, A_\alpha = c\left(\bar{L} - L_1^\Omega\right) \\
&A_\beta = \frac{c\left(\bar{L} - L_1^\Omega\right)}{3}, B_3^\Psi = B_\beta = 4\sqrt{\frac{c\left(\bar{L} - L_1^\Omega\right)}{3}} \\
&\underline{\text{pareto frontier}} \\
&u_\Psi = \frac{8c^{3/2}\left(\bar{L} - u_\Omega\right)^{3/2}\sqrt{3}}{9}, 0 \le u_\Omega \le \bar{L}
\end{aligned}} \tag{6}$$

2.COMPETITIVE EQUILIBRIA

There are three commodities, since good L is only available in the first period, good A is only available in the second period, and good B is only available in the third period.

1.NAME THE PRICE OF EACH GOOD

$p =$ price of commodity A
w=price of commodity L . Normalize $w = 1$
q=price of commodity B

2. DEFINE CONSUMER INCOMES

$$M_\Psi = \bar{L} + \Pi_\alpha, M_\Omega = (1-\tau)\Pi_\beta \tag{7}$$

3. SOLVE THE OPTIMIZATION PROBLEMS OF CONSUMERS

$\max u_\Psi = A_2^\Psi B_3^\Psi$, subject to $pA_2^\Psi + qB_3^\Psi \le M_\Psi, A_2^\Psi \ge 0, B_3^\Psi \ge 0$
$\max u_\Omega = L_1^\Omega$, subject to $0 \le L_1^\Omega \le M_\Omega$

The solutions are

$$\left(A_2^\Psi, B_3^\Psi\right) = \left(\frac{M_\Psi}{2p}, \frac{M_\Psi}{2q}\right) \tag{8}$$

$$L_1^\Omega = M_\Omega \tag{9}$$

4.1. SOLVE THE OPTIMIZATION PROBLEMS OF FIRM α

$\max \Pi_\alpha = pA_\alpha - L_\alpha - t_\alpha L_\alpha$
$L_\alpha \ge 0, A_\alpha = cL_\alpha$

The solution is

$$\left(L_\alpha, A_\alpha, \Pi_\alpha\right) = \begin{cases} (\infty,\infty,\infty) & \text{if} \quad p > (1+t_\alpha)/c \\ (L_\alpha, cL_\alpha, 0) & \text{if} \quad p = (1+t_\alpha)/c \\ (0,0,0) & \text{if} \quad p < (1+t_\alpha)/c \end{cases} \tag{10}$$

4.2. SOLVE THE OPTIMIZATION PROBLEMS OF FIRM β

$\max\ (1-\tau)\Pi_\beta = (1-\tau)\left(qB_\beta - pA_\beta + s_\beta A_\beta\right)$
$A_\beta \ge 0, B_\beta = 4\sqrt{A_\beta}$

The solution is

$$\left(A_\beta, B_\beta, \Pi_\beta\right) = \begin{cases} (\infty,\infty,\infty) & \text{if} \quad p \le s_\beta \\ \left(\dfrac{4q^2}{\left(p-s_\beta\right)^2}, \dfrac{8q}{\left(p-s_\beta\right)}, \dfrac{4q^2}{\left(p-s_\beta\right)}\right) & \text{if} \quad p > s_\beta \end{cases} \tag{11}$$

5. SOLVE THE EQUILIBRIUM CONDITIONS

$$\begin{aligned} & L_1^{\Omega} + L_\alpha = \bar{L} \\ & A_2^{\Psi} + A_\beta = A_\alpha \\ & B_3^{\Psi} = B_\beta \end{aligned} \tag{12}$$

$$\boxed{\begin{aligned}
&\boxed{\text{competitive equilibrium}} \\
&p = \frac{1+t_\alpha}{c}, q = \frac{\sqrt{\left(1+t_\alpha - cs_\beta\right)\bar{L}}}{4\sqrt{c}}, \Pi_\alpha = 0, \Pi_\beta = \frac{\bar{L}}{4} \\
&L_\alpha = \frac{\bar{L}\left(3p-2s_\beta\right)}{4cp\left(p-s_\beta\right)}, A_\alpha = \frac{\bar{L}\left(3p-2s_\beta\right)}{4p\left(p-s_\beta\right)} \\
&A_\beta = \frac{\bar{L}}{4\left(p-s_\beta\right)} = \frac{c\bar{L}}{4\left(1+t_\alpha - cs_\beta\right)}, B_\beta = 2\sqrt{\frac{\bar{L}}{\left(p-s_\beta\right)}} \\
&L_1^{\Omega} = \frac{\left(4cp^2 - 4cps_\beta - 3p + 2s_\beta\right)\bar{L}}{4\left(p-s_\beta\right)cp} = \frac{\bar{L}\left(3\left(1+t_\alpha\right) - 2cs_\beta\right)}{4\left(1+t_\alpha - cs_\beta\right)\left(1+t_\alpha\right)} \\
&A_2^{\Psi} = \frac{\bar{L}}{2p} = \frac{c\bar{L}}{2\left(1+t_\alpha\right)}, B_3^{\Psi} = \frac{2\sqrt{\bar{L}}}{\sqrt{p-s_\beta}} = \frac{2\sqrt{c\bar{L}}}{\sqrt{1+t_\alpha - cs_\beta}}
\end{aligned}} \tag{13}$$

$$\boxed{\begin{aligned}
&\boxed{\begin{aligned}&\text{conditions on the parameters} \\ &\text{necessary and sufficient for existence of equilibrium}\end{aligned}} \\
&\tau = \frac{3cs_\beta t_\alpha + cs_\beta - 3t_\alpha{}^2 - 3t_\alpha}{\left(1+t_\alpha - cs_\beta\right)\left(1+t_\alpha\right)} \text{(government budget constraint)} \\
&\frac{3t_\alpha\left(t_\alpha+1\right)}{\left(3t_\alpha+1\right)c} \le s_\beta < \frac{4t_\alpha{}^2 + 5t_\alpha + 1}{2c\left(2t_\alpha+1\right)}
\end{aligned}} \tag{14}$$

3. WHEN ARE COMPETITIVE ALLOCATIONS EFFICIENT?

By (6) at any efficient allocation $A_2^{\Psi} = 2A_\beta$ .Imposing the same condition on the equilibrium allocations we obtain, by (13), $\frac{\bar{L}}{2p} = 2\frac{\bar{L}}{4\left(p-s_\beta\right)}$,i.e.

$$s_\beta = 0 \tag{15}$$

By (15) and (14)

$$t_\alpha = 0 = \tau \tag{16}$$

$$\boxed{\begin{array}{l} \text{efficient competitive equilibrium} \\ t_\alpha = 0 = s_\beta = \tau \\ p = \frac{1}{c}, q = \frac{\sqrt{\bar{L}}}{4\sqrt{c}}, \Pi_\alpha = 0, \Pi_\beta = \frac{\bar{L}}{4} \\ L_\alpha = \frac{3\bar{L}}{4}, A_\alpha = \frac{3c\bar{L}}{4} \\ A_\beta = \frac{c\bar{L}}{4}, B_\beta = 2\sqrt{c\bar{L}} \\ L_1^\Omega = \frac{\bar{L}}{4} \\ A_2^\Psi = \frac{c\bar{L}}{2}, B_3^\Psi = 2\sqrt{c\bar{L}} \end{array}} \tag{17}$$

PROBLEM 2

THE ECONOMY

- Two goods, α and χ, written in this order. Good α is a public good; good χ is a private good.
- Two consumers,1 and 2.
- One firm.

Consumer 1

- Consumption set R_+^2
- Endowment vector $\omega_1 = \overbrace{[0,1]}^{\alpha,\chi}$
- Profit share $\theta_1 \geq 0$
- Utility function $u_1 = X_1 + \log(A)$

Consumer 2

- Consumption set R_+^2
- Endowment vector $\omega_2 = \overbrace{[0,\kappa]}^{\alpha,\chi}, \kappa > 1$
- Profit share $\theta_2 = 1 - \theta_1 \geq 0$
- Utility function $u_2 = X_2 + \rho \log(A), \rho > 1$

The firm produces good α out of good χ with technology described by the production function

$$\hat{A} = 2\hat{X}$$

QUESTIONS

Answer the following questions for all allowed parameter values

- Compute all efficient allocations. Compute and draw the Pareto frontier and the utility possibility set.
- Compute all competitive equilibria.
- For which values of the parameters, if any, are competitive equilibria efficient?

answers to problem 2

1.EFFICIENT ALLOCATIONS

The set of feasible allocations is the convex set

$$S = \left\{ \left(X_1, X_2, \hat{X} \right) \in R_+^3 : X_1 + X_2 + \hat{X} \leq 1 + k \right\} \tag{18}$$

The objective functions are both concave

$$u_1 = X_1 + \log(2\hat{X}), u_2 = X_2 + \rho \log(2\hat{X}) \tag{19}$$

Hence, we can compute efficient allocations by the linear SWF method, i.e., by solving the following maximization problem for all $0 \leq \alpha \leq 1$ (THIS α is a parameter, not the name of the public good).

$$\begin{aligned} &\max \ \ \mathrm{W}{=}\alpha u_2 + (1-\alpha)u_1 \\ &\text{subject to } X_1 + X_2 + \hat{X} \leq 1 + k, X_1 \geq 0, X_2 \geq 0, \hat{X} \geq 0 \end{aligned} \tag{20}$$

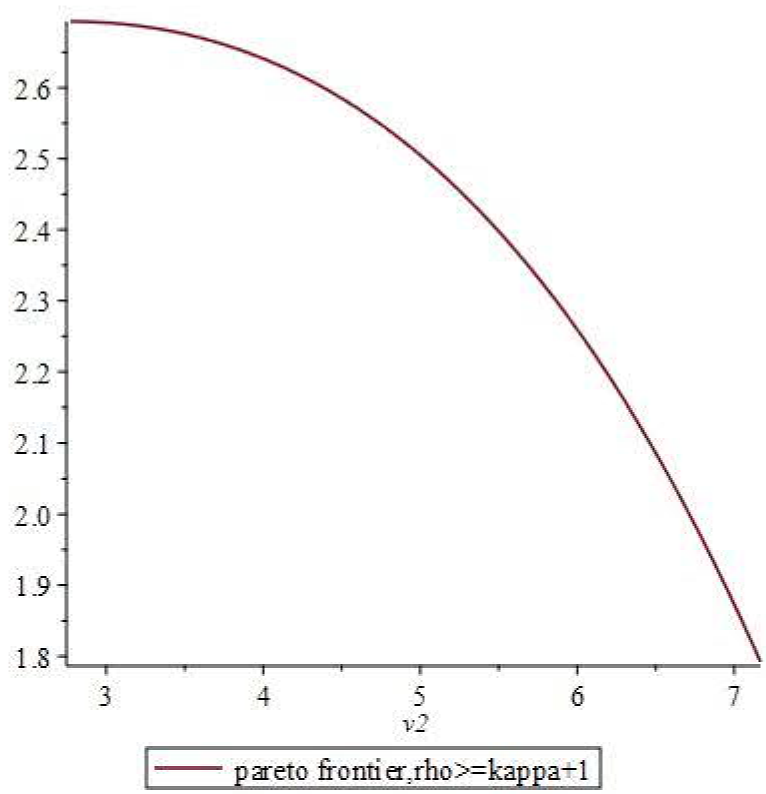
The global maxima of (20),depending on parameter values, are given by

$$\boxed{\begin{array}{l}\underline{\text{efficient allocations when } \rho \geq 1+\kappa} \\ \begin{bmatrix}\hat{X} & X_1 & X_2\end{bmatrix} = \begin{cases}\begin{bmatrix}1+\frac{\alpha\rho}{1-\alpha} & \kappa-\frac{\alpha\rho}{1-\alpha} & 0\end{bmatrix} & \text{if} \quad \alpha \leq \frac{\kappa}{\kappa+\rho} \\ \begin{bmatrix}1+\kappa & 0 & 0\end{bmatrix} & \text{if} \quad \alpha \geq \frac{1}{\kappa+\rho}\end{cases}\end{array}} \tag{21}$$

Eliminating α from (21) we obtain

$$\boxed{\begin{array}{l}\text{efficient allocations when } \rho \geq 1+\kappa \\ 1 \leq \hat{X} \leq 1+\kappa, X_1 = 1+\kappa-\hat{X}, X_2 = 0\end{array}} \tag{22}$$

$$\boxed{\begin{array}{l}\text{pareto frontier when } \rho \geq 1+\kappa \\ u_1 = 1+\kappa+\frac{u_2}{\rho}-\frac{1}{2}e^{\frac{u_2}{\rho}} \\ \rho\log 2 \leq u_2 \leq \rho\ln(2+2\kappa)\end{array}} \tag{23}$$

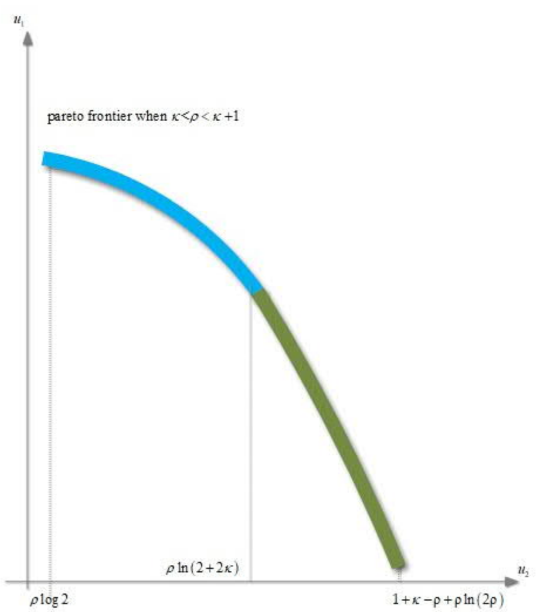


$$\boxed{\begin{array}{l}\text{efficient allocations when } \kappa < \rho < 1+\kappa \\ \begin{bmatrix}\hat{X} & X_1 & X_2\end{bmatrix} = \begin{cases} \begin{bmatrix}1+\dfrac{\alpha\rho}{1-\alpha} & \kappa-\dfrac{\alpha\rho}{1-\alpha} & 0\end{bmatrix} & \text{if} \quad \alpha \le \dfrac{\kappa}{\kappa+\rho} \\ \begin{bmatrix}1+\kappa & 0 & 0\end{bmatrix} & \text{if} \quad \dfrac{\kappa}{\kappa+\rho} \le \alpha \le \dfrac{1}{2+\kappa-\rho} \\ \begin{bmatrix}\rho+\dfrac{1-\alpha}{\alpha} & 0 & 2+\kappa-\rho-\dfrac{1}{\alpha}\end{bmatrix} & \text{if} \quad \alpha \ge \dfrac{1}{2+\kappa-\rho} \end{cases}\end{array}} \tag{24}$$

Eliminating α from (24) we obtain

$$\boxed{\begin{array}{l}\text{efficient allocations when } \kappa < \rho < 1+\kappa \\ 1 \le \hat{X} \le 1+\kappa, X_1 = 1+\kappa-\hat{X}, X_2 = 0, \text{and} \\ \rho \le \hat{X} \le 1+\kappa, X_1 = 0, X_2 = 1+\kappa-\hat{X}\end{array}} \tag{25}$$

$$\boxed{\begin{array}{l}\text{pareto frontier when } \kappa < \rho < 1+\kappa \\ u_1 = 1+\kappa+\dfrac{u_2}{\rho}-\dfrac{1}{2}e^{\frac{u_2}{\rho}}, \text{when } \rho\log 2 \le u_2 \le \rho\ln(2+2\kappa) \\ u_2 = 1+\kappa+\rho u_1-\dfrac{1}{2}e^{u_1} \text{ when} \\ \rho\ln(2+2\kappa) \le u_2 \le 1+\kappa-\rho+\rho\ln(2\rho)\end{array}} \tag{26}$$

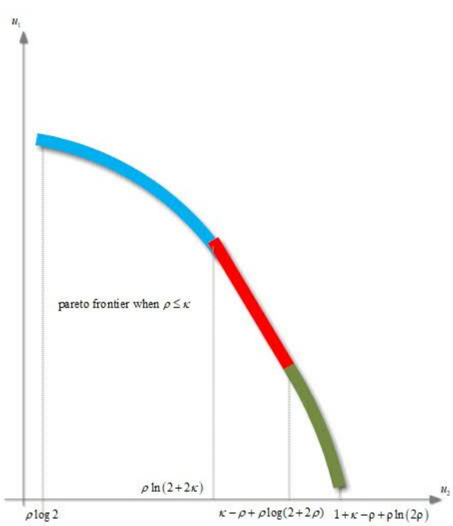


$$\boxed{\begin{array}{l}\text{efficient allocations when } \rho \le \kappa \\ \begin{bmatrix}\hat{X} & X_1 & X_2\end{bmatrix} = \begin{cases} \begin{bmatrix} 1+\dfrac{\alpha\rho}{1-\alpha} & \kappa-\dfrac{\alpha\rho}{1-\alpha} & 0 \end{bmatrix} & \text{if} \quad \alpha < \dfrac{1}{2} \\ \begin{bmatrix} 1+\rho & X_1 & \kappa-\rho-X_1 \end{bmatrix}, 0 \le X_1 \le \kappa-\rho & \text{if} \quad \alpha = \dfrac{1}{2} \\ \begin{bmatrix} \rho+\dfrac{1-\alpha}{\alpha} & 0 & 2+\kappa-\rho-\dfrac{1}{\alpha} \end{bmatrix} & \text{if} \quad \alpha > \dfrac{1}{2} \end{cases}\end{array}} \tag{27}$$

Eliminating α from (27) we obtain

$$\boxed{\begin{array}{l}\text{efficient allocations when } \rho \le \kappa \\ 1 \le \hat{X} < 1+\rho, X_1 = 1+\kappa-\hat{X}, X_2 = 0, \text{and} \\ \rho \le \hat{X} < 1+\rho, X_1 = 0, X_2 = 1+\kappa-\hat{X}, \text{and} \\ \hat{X} = 1+\rho, X_1+X_2 = \kappa-\rho, 0 \le X_1 \le \kappa-\rho \end{array}} \tag{28}$$

$$\boxed{\begin{array}{l}\text{pareto frontier when } \rho \le \kappa \\ u_1 = 1+\kappa+\dfrac{u_2}{\rho}-\dfrac{1}{2}e^{\frac{u_2}{\rho}}, \text{when } \rho\log 2 \le u_2 \le \rho\ln\left(2+2\kappa\right) \\ u_1+u_2 = \kappa-\rho+\left(1+\rho\right)\log(2+2\rho), \text{when} \\ \rho\ln\left(2+2\kappa\right) \le u_2 \le \kappa-\rho+\rho\log(2+2\rho) \\ u_2 = 1+\kappa+\rho u_1-\dfrac{1}{2}e^{u_1} \text{ when} \\ \kappa-\rho+\rho\log(2+2\rho) \le u_2 \le 1+\kappa-\rho+\rho\ln\left(2\rho\right) \end{array}} \tag{29}$$



2.COMPETITIVE EQUILIBRIA

1.NAME THE PRICE OF EACH GOOD

$\begin{array}{l} p = \text{price of commodity } \alpha \\ \text{w=price of commodity } \chi \end{array}$. Normalize $w = 1$

2. DEFINE CONSUMER INCOMES

$$M_1 = 1 + \theta_1 \Pi, M_2 = \kappa + (1 - \theta_1)\Pi \tag{30}$$

3. SOLVE THE OPTIMIZATION PROBLEMS OF THE FIRM

$\max \Pi = p\hat{X} - \hat{X}$, subject to $\hat{X} \geq 0, \hat{A} = 2\hat{X}$

variables: $\hat{X}, \hat{A}$

parameters: p

conditions on parameters: $p > 0$

The solution is

$$\begin{bmatrix} \hat{X} & \hat{A} & \Pi \end{bmatrix} = \begin{cases} (\infty, \infty, \infty) & \text{if} \quad p > 1/2 \\ (\hat{X}, 2\hat{X}, 0) & \text{if} \quad p = 1/2 \\ (0, 0, 0) & \text{if} \quad p < 1/2 \end{cases} \tag{31}$$

Since $p > 1/2$ is never going to be an equilibrium price, we set

$$p \leq 1/2, \Pi = 0, M_1 = 1, M_2 = \kappa \tag{32}$$

4.1. SOLVE THE OPTIMIZATION PROBLEM OF CONSUMER 1

$\max u_1 = X_1 + \log(A_1 + A_2)$, subject to $pA_1 + X_1 \leq 1, A_1 \geq 0, X_1 \geq 0$

variables: X_1, A_1

parameters: p, A_2

conditions on parameters: $p > 0, A_2 \geq 0$

The solutions are

demand functions of consumer 1

$$[X_1 \quad A_1] = \begin{cases} \left[pA_2 \quad \frac{1}{p} - A_2\right] & \text{if} \quad A_2 \le 1/p \\ [1 \quad 0] & \text{if} \quad A_2 \ge 1/p \end{cases} \tag{33}$$

4.2. SOLVE THE OPTIMIZATION PROBLEM OF CONSUMER 2

max $u_2 = X_2 + \rho \log(A_1 + A_2)$, subject to $pA_2 + X_2 \le \kappa, A_2 \ge 0, X_2 \ge 0$
variables: X_2, A_2
parameters: p, A_1, κ
conditions on parameters: $p > 0, A_1 \ge 0, \kappa > 1$

The solutions are

demand functions of consumer 2 when $\rho \le \kappa$

$$[X_2 \quad A_2] = \begin{cases} \left[\kappa - \rho + pA_1 \quad \frac{\rho}{p} - A_1\right] & \text{if} \quad A_1 \le \frac{\rho}{p} \\ [\kappa \quad 0] & \text{if} \quad A_1 \ge \frac{\rho}{p} \end{cases} \tag{34}$$

demand functions of consumer 2 when $\rho > \kappa$

$$[X_2 \quad A_2] = \begin{cases} \left[0 \quad \frac{\kappa}{p}\right] & \text{if} \quad A_1 \le \frac{\rho - \kappa}{p} \\ \left[\kappa - \rho + pA_1 \quad \frac{\rho}{p} - A_1\right] & \text{if} \quad \frac{\rho - \kappa}{p} \le A_1 \le \frac{\rho}{p} \\ [\kappa \quad 0] & \text{if} \quad A_1 \ge \frac{\rho}{p} \end{cases} \tag{35}$$

5. SOLVE THE EQUILIBRIUM CONDITIONS

$$\begin{aligned} A_1 + A_2 &= \hat{A} = 2\hat{X} \\ X_1 + X_2 + \hat{X} &= 1 + \kappa \end{aligned} \tag{36}$$

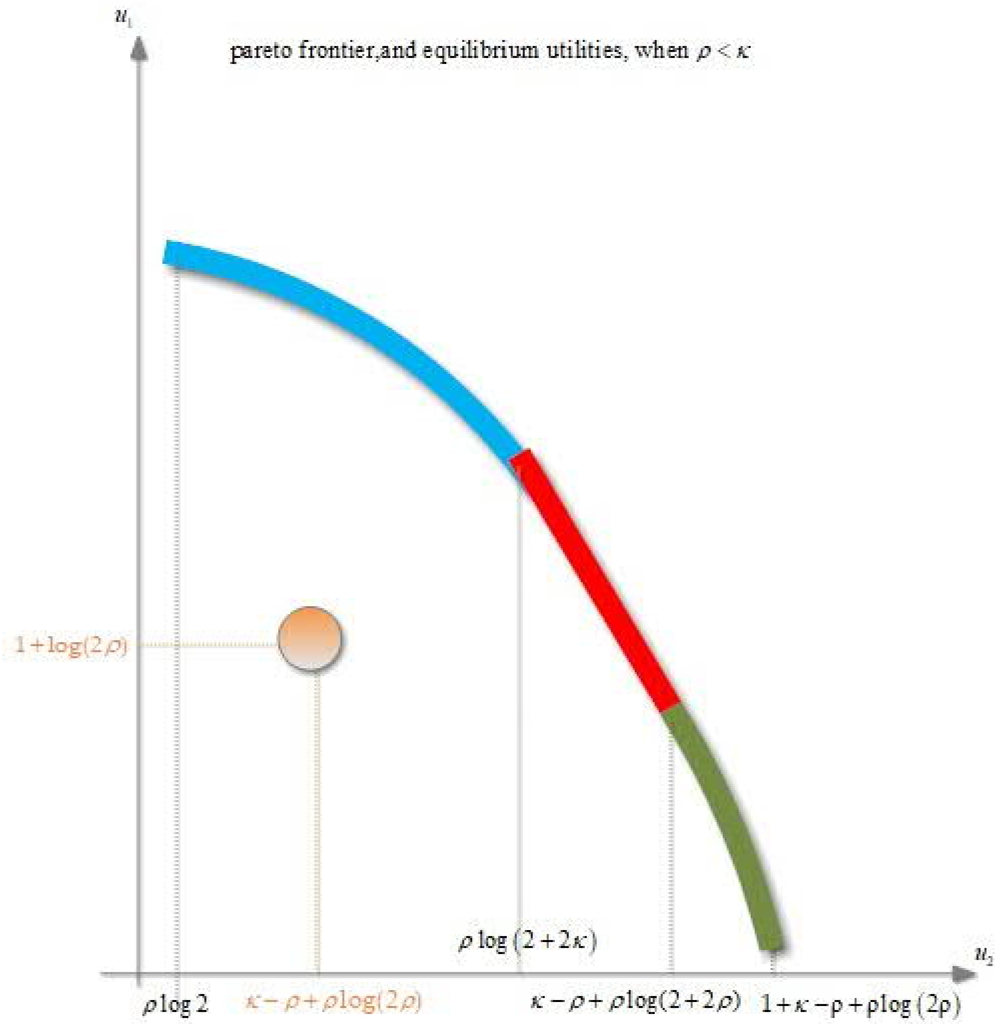
Equilibrium prices and quantities are given by

$$\boxed{\begin{array}{l} \underline{\text{competitive equilibrium when } \rho \le \kappa} \\ p = \frac{1}{2}, \Pi = 0, \hat{X} = \rho, \hat{A} = 2\rho \\ A_1 = 0, X_1 = 1, A_2 = 2\rho, X_2 = \kappa - \rho \\ u_1 = 1 + \log(2\rho), u_2 = \kappa - \rho + \rho\log(2\rho) \end{array}} \tag{37}$$

$$\boxed{\begin{array}{l} \underline{\text{competitive equilibrium when } \rho > \kappa} \\ p = \frac{1}{2}, \Pi = 0, \hat{X} = \kappa, \hat{A} = 2\kappa \\ A_1 = 0, X_1 = 1, A_2 = 2\kappa, X_2 = 0 \\ u_1 = 1 + \log(2\kappa), u_2 = \rho\log(2\kappa) \end{array}} \tag{38}$$

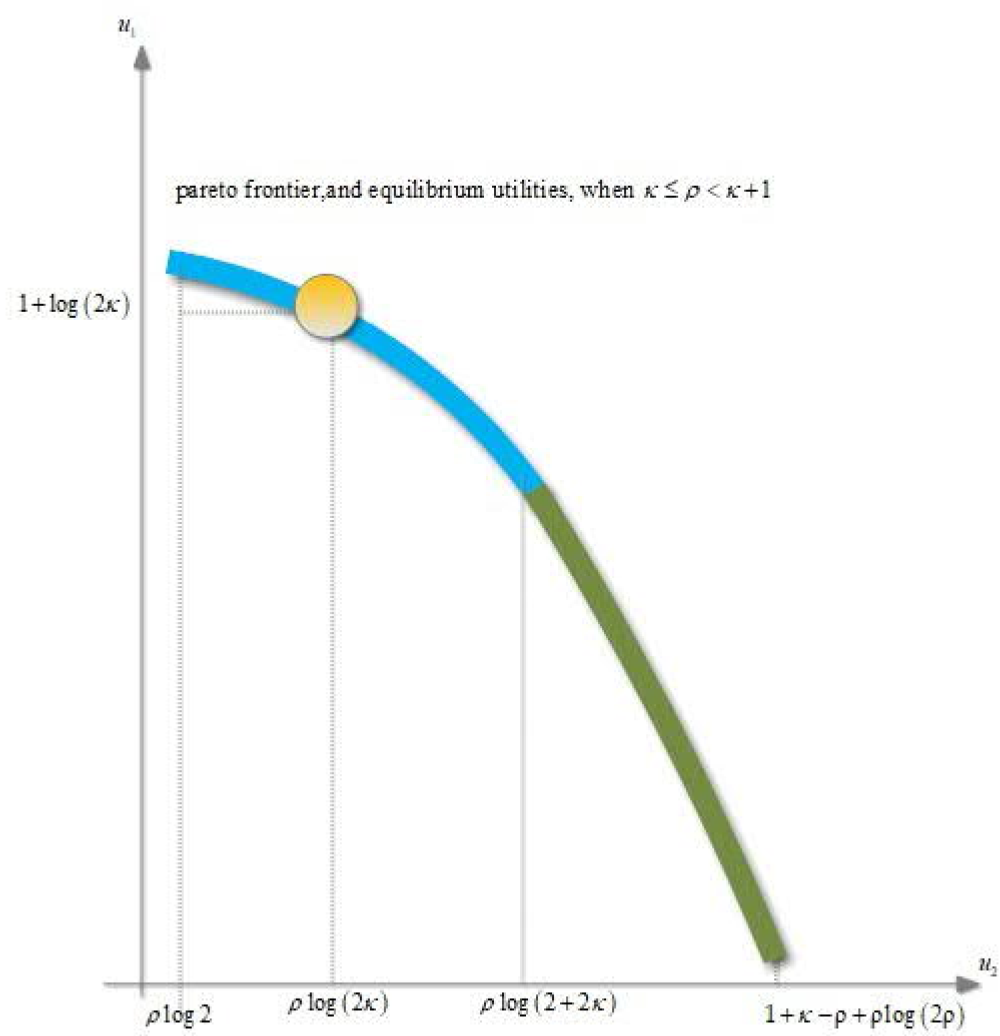
3.WHEN ARE COMPETITIVE ALLOCATIONS EFFICIENT?

When $\rho < \kappa$ we must decide whether the allocation in (37) is one of the allocations in (28);by inspection, the answer is negative, hence when $\rho < \kappa$ equilibria are inefficient.



When $\rho = \kappa$ we must decide whether the allocation in (37) is one of the allocations in (28);by inspection, the answer is positive, hence when $\rho = \kappa$ equilibria are efficient.

When $\rho > \kappa$ we must decide whether the allocation in (38) is one of the allocations in (22) or (25);by inspection, the answer is positive, hence when $\rho > \kappa$ equilibria are efficient.



PROBLEM 3

THE ECONOMY

- Four goods, A, B, L, K
- One consumer
- Two firms, α and β

Consumer

- Consumption set R^4_+
- Endowment vector $\omega = \overbrace{[0, 0, \bar{L}, \bar{K}]}^{A,B,L,K}$
- Utility function $u = \alpha \log(A) + (1-\alpha)\log(B), 0 < \alpha < 1$

Firm α produces good A out of goods K, L with production function

$$\hat{A} = \min\{K_A, \rho L_A\}, \rho > 0$$

where K_A, L_A are the quantities of goods K, L, respectively, used as inputs in the production of good A, and $\hat{A}$ is the quantity produced of good A.

Firm β produces good B out of goods K, L with technology described by the production function

$$\hat{B} = 2\sqrt{K_B L_B}$$

where K_B, L_B are the quantities of goods K, L, respectively, used as inputs in the production of good B, and $\hat{B}$ is the quantity produced of good B.

QUESTIONS

Answer the following questions for all allowed parameter values

- Compute all competitive equilibria.
- For which values of the parameters, if any, do competitive equilibria exist?
- Plot equilibrium prices as a function of α ,keeping all other parameters fixed
- Plot equilibrium prices as a function of $\bar{K}$,keeping all other parameters fixed

answers to problem 3

1.COMPETITIVE EQUILIBRIA

1.NAME THE PRICE OF EACH GOOD

p = price of commodity A,q=price of commodity B
w=price of commodity L,r=price of commodity K
Normalize $r = 1$

2. DEFINE CONSUMER INCOME

$$M = w\bar{L} + \bar{K} + \Pi_\alpha + \Pi_\beta \tag{39}$$

3. SOLVE THE OPTIMIZATION PROBLEM OF THE CONSUMER

max $u = \alpha \log(A) + (1-\alpha)\log(B)$,subject to $pA + qB \le M$, $A \ge 0, B \ge 0$

The solutions are

$$\left(A,B\right)=\left(\alpha\frac{M}{p},(1-\alpha)\frac{M}{q}\right) \tag{40}$$

4.1. SOLVE THE OPTIMIZATION PROBLEM OF FIRM α

$$\max \Pi_\alpha = p\hat{A}-wL_A-K_A$$
$$\text{subject to } L_A\geq 0, K_A\geq 0, \hat{A}=\min\left\{K_A,\rho L_A\right\}$$

The solution is (example 13 in the notes on optimization)

$$\left(L_A,K_A,\hat{A},\Pi_\alpha\right)=\begin{cases}\left(\infty,\infty,\infty,\infty\right) & \text{if} \quad p>1+\dfrac{w}{\rho}\\ \left(L_A,\rho L_A,\rho L_A,0\right) & \text{if} \quad p=1+\dfrac{w}{\rho}\\ \left(0,0,0,0\right) & \text{if} \quad p<1+\dfrac{w}{\rho}\end{cases} \tag{41}$$

4.2. SOLVE THE OPTIMIZATION PROBLEM OF FIRM β

$$\max \Pi_\beta = q\hat{B}-wL_B-K_B$$
$$\text{subject to } L_B\geq 0, K_B\geq 0, \hat{B}=2\sqrt{K_BL_B}$$

The solution is (example 6 in the notes on optimization)

$$\left(L_B,K_B,\hat{B},\Pi_\beta\right)=\begin{cases}\left(\infty,\infty,\infty,\infty\right) & \text{if} \quad q^2>w\\ \left(\dfrac{K_B}{w},K_B,2\dfrac{K_B}{q},0\right) & \text{if} \quad q^2=w\\ \left(0,0,0,0\right) & \text{if} \quad q^2<w\end{cases} \tag{42}$$

5. SOLVE THE EQUILIBRIUM CONDITIONS

$$\begin{aligned} &A=\hat{A}\\ &B=\hat{B}\\ &L_A+L_B=\bar{L}\\ &K_A+K_B=\bar{K} \end{aligned} \tag{43}$$

$$\boxed{\begin{aligned} &\text{competitive equilibrium}\\ &p=1+w/\rho, w=q^2, \Pi_\alpha=0, \Pi_\beta=0\\ &q=\frac{\sqrt{\sqrt{\Delta}+(1+\alpha)\left(\bar{K}-\rho\bar{L}\right)}}{\sqrt{2\bar{L}}\sqrt{1-\alpha}}\\ &\Delta=\bar{L}^2(\alpha+1)^2\rho^2+2\bar{K}\bar{L}\left(\alpha^2-6\alpha+1\right)\rho+\bar{K}^2(\alpha+1)^2\\ &K_A=\hat{A}=\frac{\left(\rho\bar{L}+\bar{K}\right)(1+\alpha)}{4}-\frac{\sqrt{\Delta}}{4}, L_A=\frac{\hat{A}}{\rho}\\ &K_B=\frac{\sqrt{\Delta}}{4}+\left(\frac{3-\alpha}{4}\right)\bar{K}-\left(\frac{1+\alpha}{4}\right)\rho\bar{L}, L_B=\frac{K_B}{w}, \hat{B}=2\frac{K_B}{q} \end{aligned}} \tag{44}$$

$$\boxed{\begin{aligned} &\text{competitive equilibrium when } \bar{K}=\rho\bar{L}\\ &p=2, w=\rho, \Pi_\alpha=0, \Pi_\beta=0, q=\sqrt{\rho}\\ &\Delta=\left(2(1-\alpha)\rho\bar{L}\right)^2\\ &K_A=\hat{A}=\alpha\rho\bar{L}, L_A=\alpha\bar{L}\\ &K_B=(1-\alpha)\rho\bar{L}, L_B=(1-\alpha)\bar{L}, \hat{B}=2(1-\alpha)\bar{L}\sqrt{\rho} \end{aligned}} \tag{45}$$

PROBLEM 4

THE ECONOMY

- Two goods,1 and 2, written in this order.
- Two consumers, A and B.

Consumer A

- Consumption set $X_A=\{(A_1,A_2):A_1+A_2\geq 2, A_1\geq 0, A_2\geq 0\}$
- Endowment vector $\omega_A=[0,2]$
- Utility function $u_A=A_1A_2$

Consumer B

- Consumption set $X_B = R_+^2$
- Endowment vector $\omega_B = [\kappa, 0], \kappa > 0$
- Utility function $u_B = B_1 B_2$

QUESTIONS

Answer the following questions for all allowed parameter values

- Compute all competitive equilibria.
- For which values of the parameters, if any, do competitive equilibria exist?

answers to problem 4

1.COMPETITIVE EQUILIBRIA

1.NAME THE PRICE OF EACH GOOD

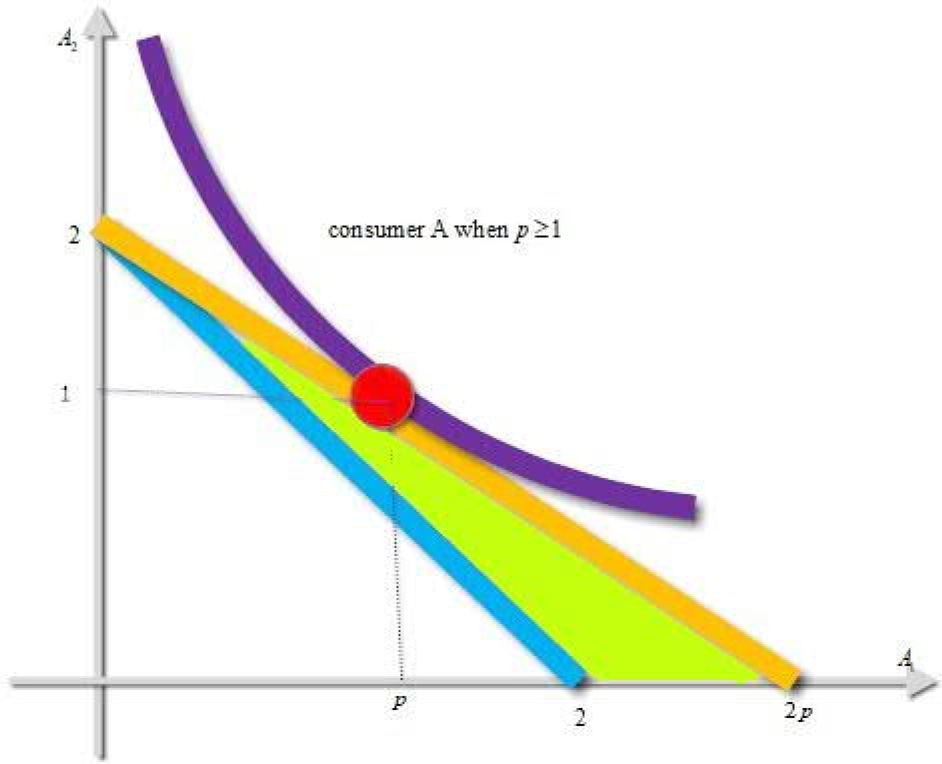
$p = \text{price of commodity } 2$. Normalize the price of commodity 1 to 1

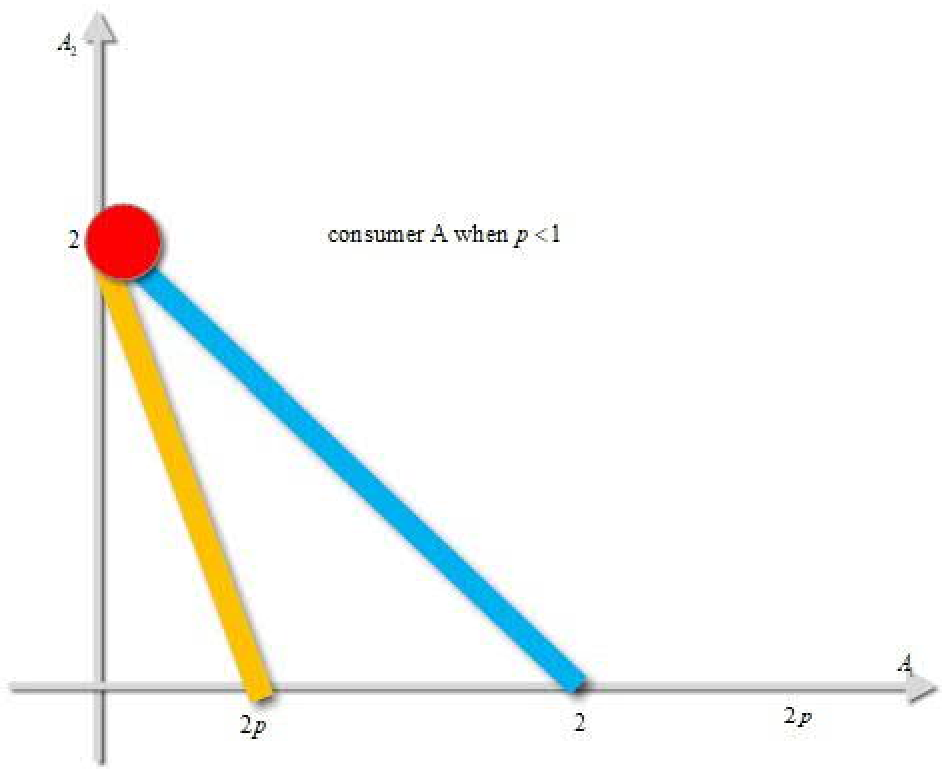
2. DEFINE CONSUMER INCOMES

$$M_A = 2p, M_B = \kappa \tag{46}$$

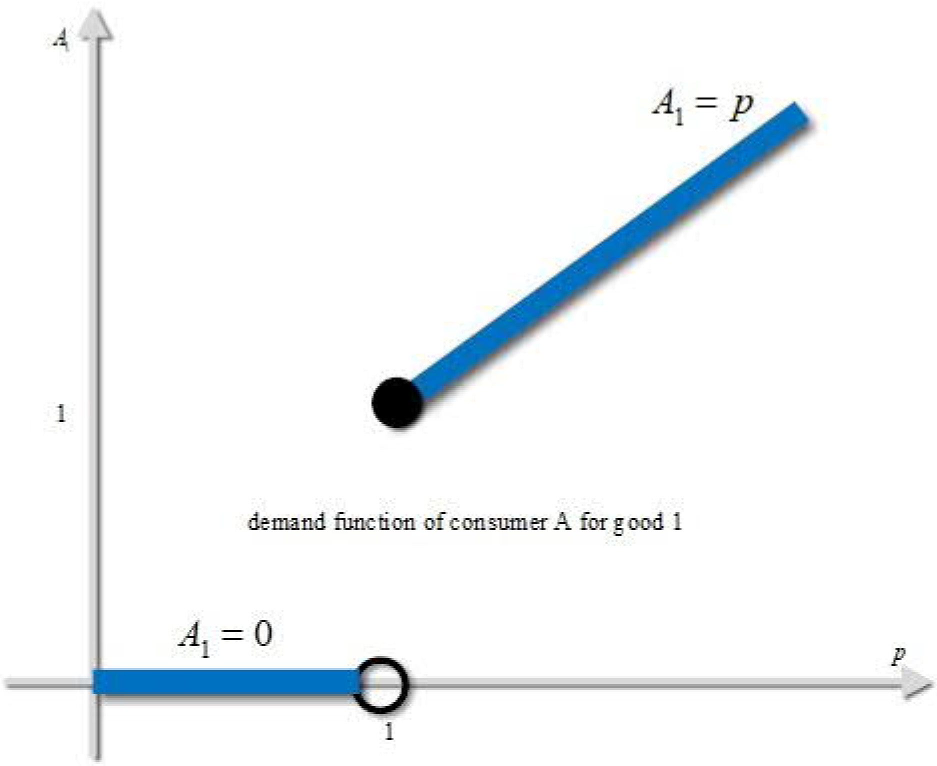
4A. SOLVE THE OPTIMIZATION PROBLEM OF CONSUMER A

$\max u_A = A_1 A_2$, subject to $A_1 + pA_2 \leq 2p, A_1 + A_2 \geq 2, A_1 \geq 0, A_2 \geq 0$
variables: A_1, A_2
parameters: p
conditions on parameters: $p > 0$


A_2
2
1
consumer A when $p \geq 1$
A_1
p
2
$2p$


A_2
2
consumer A when $p < 1$
A_1
$2p$
2
$2p$

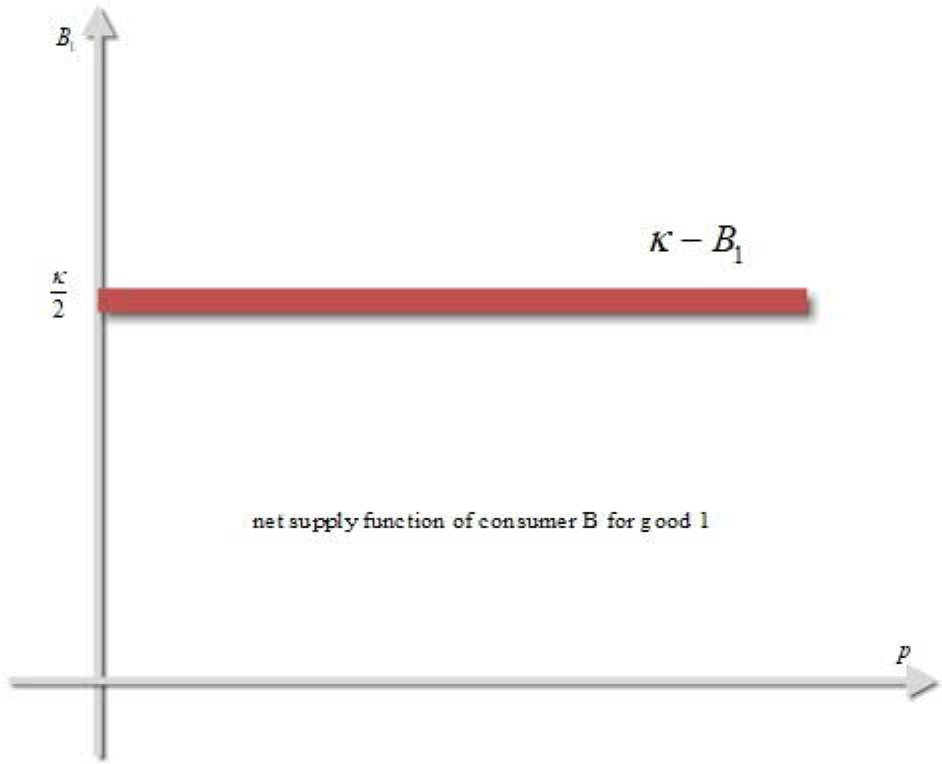
$$\boxed{\begin{array}{l}\text{demand functions of consumer A}\\ \begin{bmatrix}A_1 & A_2\end{bmatrix}=\begin{cases}\begin{bmatrix}p & 1\end{bmatrix} & \text{if} \quad p\geq 1\\ \begin{bmatrix}0 & 2\end{bmatrix} & \text{if} \quad p<1\end{cases}\end{array}} \tag{47}$$



4B. SOLVE THE OPTIMIZATION PROBLEM OF CONSUMER 8

max $u_B = B_1 B_2$,subject to $B_1 + pB_2 \leq \kappa, B_1 \geq 0, B_2 \geq 0$
variables:B_1, B_2
parameters:p, κ
conditions on parameters:$p > 0, \kappa > 0$

$$\boxed{\begin{array}{l}\text{demand functions of consumer B}\\ \begin{bmatrix}B_1 & B_2\end{bmatrix}=\begin{bmatrix}\dfrac{\kappa}{2} & \dfrac{\kappa}{2p}\end{bmatrix}\end{array}} \tag{48}$$

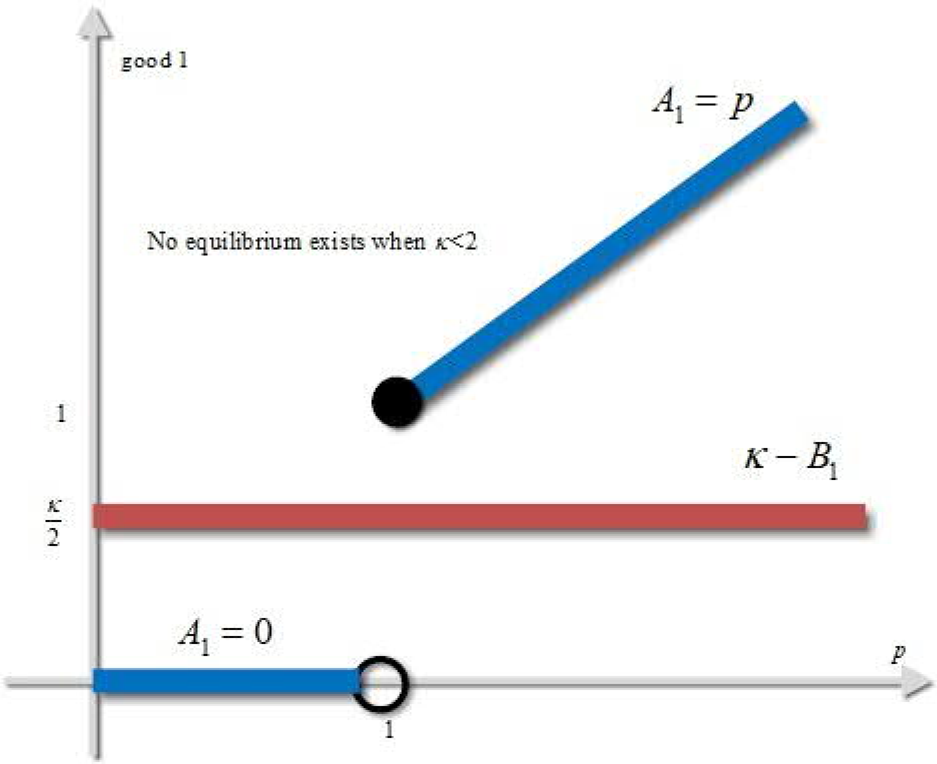


5. SOLVE THE EQUILIBRIUM CONDITIONS

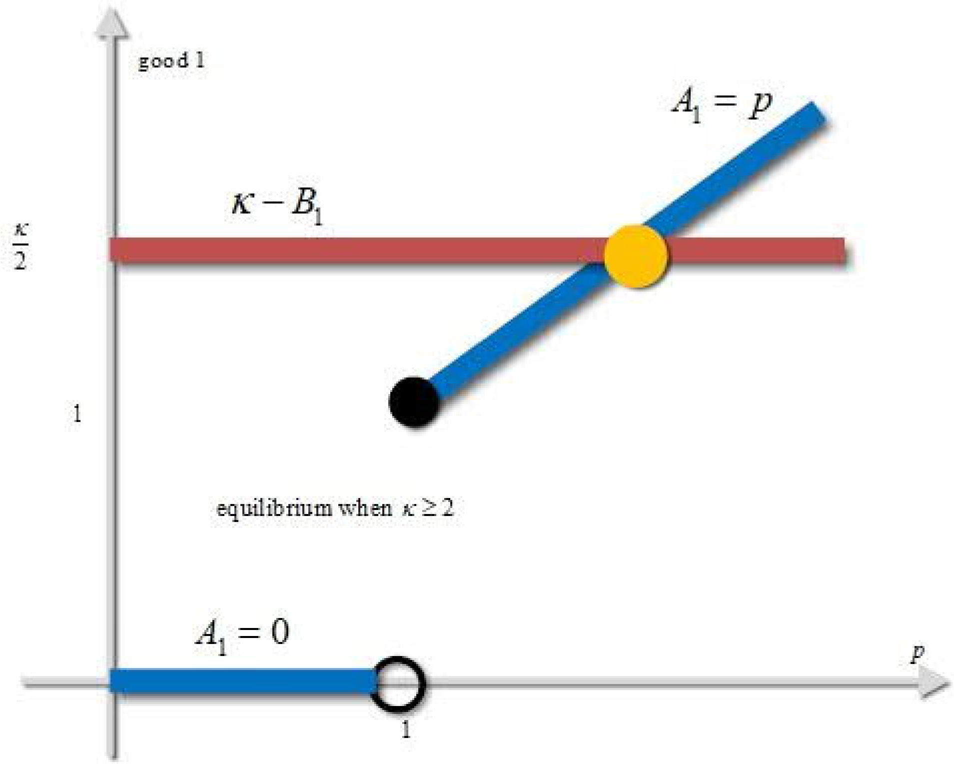
$$\begin{aligned} A_1 + B_1 &= \kappa \\ A_2 + B_2 &= 2 \end{aligned} \tag{49}$$

Equilibrium prices and quantities are given by

$$\begin{array}{|l|} \hline \text{competitive equilibrium when } \kappa < 2 \\ \hline \text{No equilibrium exists} \\ \hline \end{array} \tag{50}$$



$$\boxed{\begin{array}{l} \text{competitive equilibrium when } \kappa \geq 2 \\ p = \frac{\kappa}{2}, A = B = \begin{bmatrix} \frac{\kappa}{2} & 1 \end{bmatrix} \end{array}} \tag{51}$$



PROBLEM 5

THE ECONOMY

Consider a five-period economy with

●one consumer

●two goods, namely A and K

● Two firms, namely γ and δ

firm γ produces good A out of good K with technology described by

$$\hat{A}_2 = 2\hat{K}_1, \hat{A}_3 = \hat{K}_1 \tag{52}$$

where $\hat{K}_1$ is the quantity of good K used by firm γ as an input in the first period, and $\hat{A}_2, \hat{A}_3$ are the quantities of good A produced by firm γ in the second and third periods.

firm δ produces good A out of good K with technology described by

$$\hat{A}_4 = 2\hat{K}_3, \hat{A}_5 = \hat{K}_3 \tag{53}$$

where $\hat{K}_3$ is the quantity of good K used by firm δ as an input in the third period, and $\hat{A}_4, \hat{A}_5$ are the quantities of good A produced by firm δ in the fourth and fifth periods.

Consumer

● consumption set: all variables ≥ 0

● utility function

$$u = \sum_{t=1}^{5} \log(A_t) + \sum_{t=1}^{5} \log(K_t) \tag{54}$$

where A_t is the quantity of good A consumed by the consumer in period t, and K_t is the quantity of good K consumed by the consumer in period t

●endowment: one unit of good A in the first period only, and one unit of good K in each period.

QUESTIONS

Answer the following questions

- Compute all competitive equilibria.
- Plot equilibrium prices and quantities as functions of t

answers to problem 5

COMPETITIVE EQUILIBRIA

1.NAME THE PRICE OF EACH GOOD

$p_t =$ price of good A in period t
$r_t =$price of good K in period t
. Normalize $p_1 = 1$

2. DEFINE CONSUMER INCOME

$$M = p_1 + \sum_{t=1}^{5} r_t + \Pi_\gamma + \Pi_\delta \tag{55}$$

3. SOLVE THE OPTIMIZATION PROBLEM OF THE CONSUMER

$\max u = \sum_{t=1}^{5} \log(A_t) + \sum_{t=1}^{5} \log(K_t)$, subject to

$\sum_{t=1}^{5} p_t A_t + \sum_{t=1}^{5} r_t K_t \le M$

variables: $A_t, K_t, t = 1..5$

parameters: $p_t, r_t, t = 1..5$

conditions on parameters: $p_1 = 1, p_t > 0, r_t > 0, t = 1..5$

$$\underline{\text{consumer demand functions}}$$

$$\left(A_t, K_t\right) = \left(\frac{M}{10p_t}, \frac{M}{10r_t}\right), t = 1..5 \tag{56}$$

4Γ. SOLVE THE OPTIMIZATION PROBLEMS OF FIRM γ

$\max \Pi_\gamma = p_2 \hat{A}_2 + p_3 \hat{A}_3 - r_1 \hat{K}_1$, subject to

$\hat{A}_2 = 2\hat{K}_1, \hat{A}_3 = \hat{K}_1 \ge 0$

$$\underline{\text{supply/input demand functions of firm } \gamma}$$

$$\begin{bmatrix} \hat{K}_1 & \hat{A}_2 & \hat{A}_3 & \Pi_\gamma \end{bmatrix} = \begin{cases} \begin{bmatrix} \infty & \infty & \infty & \infty \end{bmatrix} & \text{if} \quad r_1 < p_3 + 2p_2 \\ \begin{bmatrix} \hat{K}_1 & 2\hat{K}_1 & \hat{K}_1 & 0 \end{bmatrix} & \text{if} \quad r_1 = p_3 + 2p_2 \\ \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} & \text{if} \quad r_1 > p_3 + 2p_2 \end{cases} \tag{57}$$

4Δ. SOLVE THE OPTIMIZATION PROBLEMS OF FIRM δ

$\max \Pi_\delta = p_4 \hat{A}_4 + p_5 \hat{A}_5 - r_3 \hat{K}_3$, subject to

$\hat{A}_4 = 2\hat{K}_3, \hat{A}_5 = \hat{K}_3 \ge 0$

$$\underline{\text{supply/input demand functions of firm } \delta}$$

$$\begin{bmatrix} \hat{K}_3 & \hat{A}_4 & \hat{A}_5 & \Pi_\delta \end{bmatrix} = \begin{cases} \begin{bmatrix} \infty & \infty & \infty & \infty \end{bmatrix} & \text{if} \quad r_3 < p_5 + 2p_4 \\ \begin{bmatrix} \hat{K}_3 & 2\hat{K}_3 & \hat{K}_3 & 0 \end{bmatrix} & \text{if} \quad r_3 = p_5 + 2p_4 \\ \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} & \text{if} \quad r_3 > p_5 + 2p_4 \end{cases} \tag{58}$$

5. SOLVE THE EQUILIBRIUM CONDITIONS

$$\boxed{\text{equilibrium conditions}}$$
$$\begin{aligned} & A_1 = 1, A_t = \hat{A}_t, t = 2..5 \\ & K_1 + \hat{K}_1 = 1, K_3 + \hat{K}_3 = 1 \\ & K_2 = 1, K_4 = 1, K_5 = 1 \end{aligned} \tag{59}$$

$$\boxed{\begin{aligned} & \boxed{\text{competitive equilibrium}} \\ & p_2 = \frac{3}{4}, p_3 = \frac{3}{2}, p_4 = \frac{3}{4}, p_5 = \frac{3}{2}, \Pi_\gamma = \Pi_\delta = 0 \\ & r_1 = 3, r_2 = 1, r_3 = 3, r_4 = 1, r_5 = 1 \\ & K_1 = \frac{1}{3}, K_2 = 1, K_3 = \frac{1}{3}, K_4 = 1, K_5 = 1 \\ & A_1 = 1, A_2 = \frac{4}{3}, A_3 = \frac{2}{3}, A_4 = \frac{4}{3}, A_5 = \frac{2}{3} \\ & \hat{K}_1 = \hat{K}_3 = \frac{2}{3}, \hat{A}_2 = \frac{4}{3}, \hat{A}_3 = \frac{2}{3}, \hat{A}_4 = \frac{4}{3}, \hat{A}_5 = \frac{2}{3} \end{aligned}} \tag{60}$$

Plotting the equilibrium values of the variables as functions of time we observe endogenous cycles

PROBLEM 6

THE ECONOMY

- Two goods, A and X ,written in this order.
- One consumer
- One firm.

The Consumer

- Consumption set R_+^2
- Endowment vector $\omega = [0, \bar{X}]$
- Profit share $\theta = 1$
- Utility function $u = \log A + \log X$

The firm produces good A out of good X with technology described by the production function

$$\hat{A} = \begin{cases} \hat{X}^2 & \text{if} \quad 0 \le \hat{X} \le F \\ F^2 & \text{if} \quad \hat{X} \ge F \end{cases} \tag{61}$$

Parameters: $F, \bar{X}$ .Conditions on parameters: $0 \le F < \bar{X}$

QUESTIONS

Answer the following questions for all allowed parameter values

- Compute all competitive equilibrium allocations E
- Compute all efficient allocations P
- First welfare theorem: For which parameter values, if any, is it true that $E \subset P$?
- Compute the set of decentralizable efficient allocations $P \bigcap E$
- Second welfare theorem: For which parameter values, if any, is it true that all efficient allocations are decentralizable, i.e., that $P = E$?

answers to problem 6

1.COMPETITIVE EQUILIBRIA

1.NAME THE PRICE OF EACH GOOD

$\begin{array}{l} p = \text{price of good A} \\ w\text{=price of good X} \end{array}$. Normalize $p = 1$

2. DEFINE CONSUMER INCOME

$$M = w\bar{X} + \Pi \quad (62)$$

3. SOLVE THE OPTIMIZATION PROBLEM OF THE CONSUMER

max $u = \log A + \log X$,subject to

$A + wX \le M$

variables:A, X

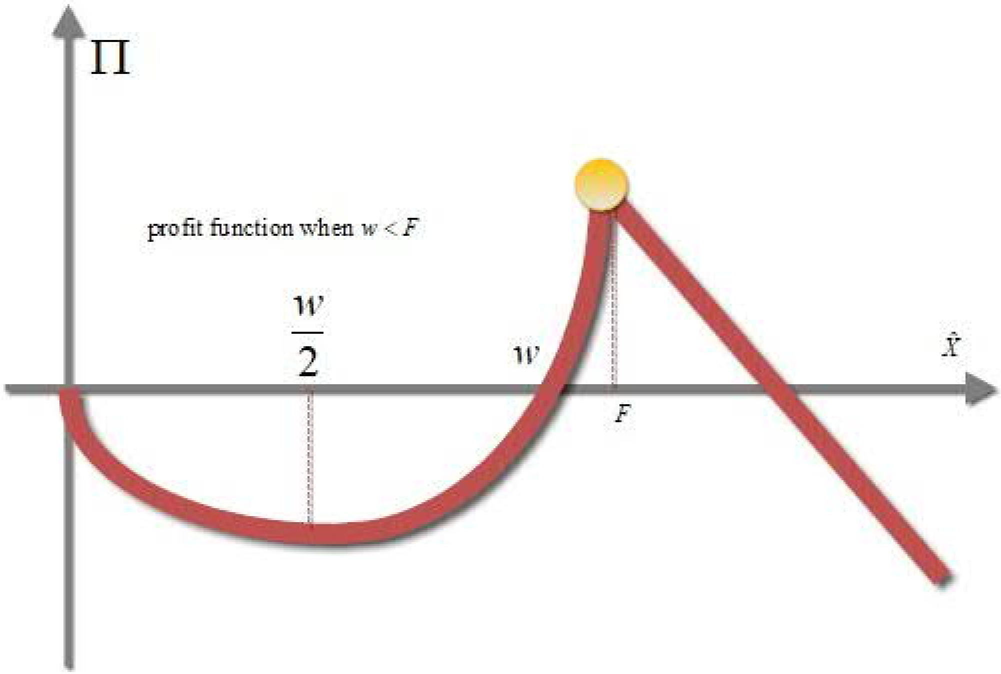
parameters:p, M

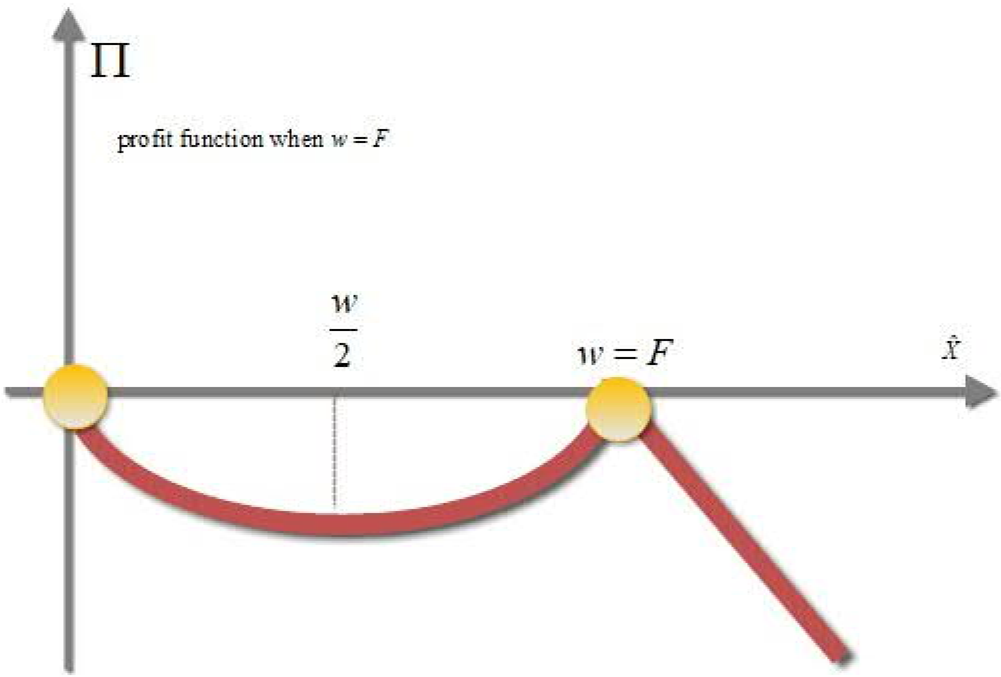
conditions on parameters:$p > 0, M > 0$

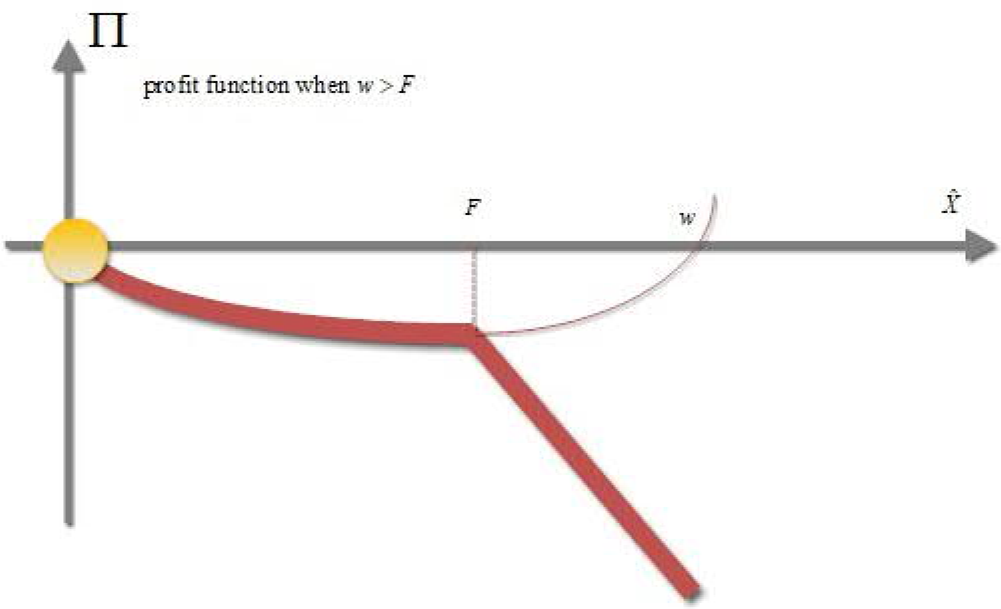
$$\underline{\text{consumer demand functions}}$$
$$(A, X) = \left(\frac{M}{2}, \frac{M}{2w}\right) \quad (63)$$

4. SOLVE THE OPTIMIZATION PROBLEMS OF THE FIRM

max $\Pi = \hat{A} - w\hat{X}$,subject to $\hat{X} \ge 0$, and to (61)

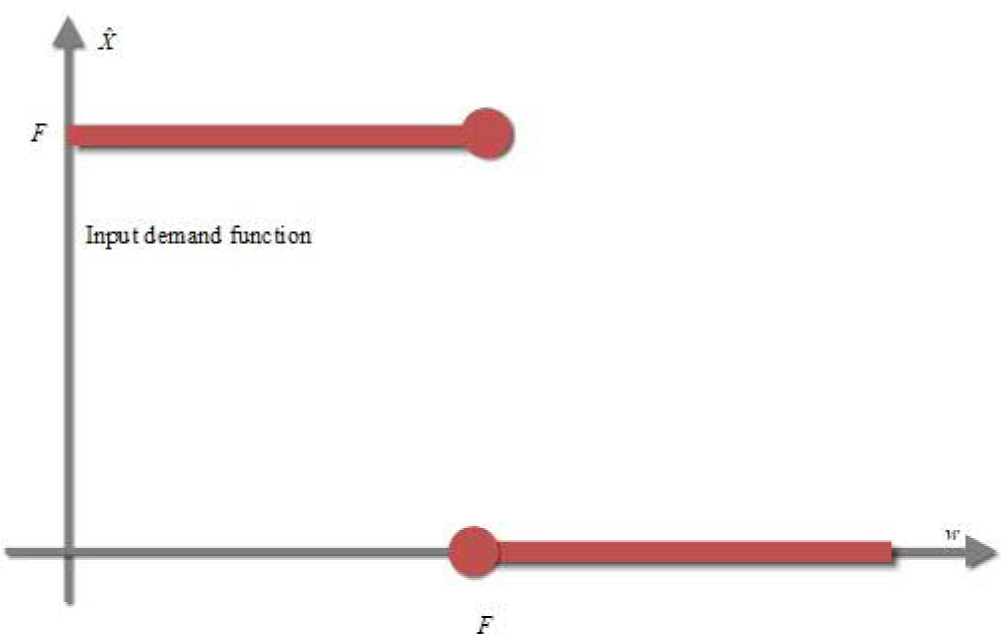

Π
profit function when $w < F$
$\frac{w}{2}$
w
F
$\hat{X}$


Π
profit function when $w = F$
$\frac{w}{2}$
$w = F$
$\hat{X}$



supply/input demand functions of the firm

$$\begin{bmatrix}\hat{X} & \hat{A} & \Pi\end{bmatrix} = \begin{cases} \begin{bmatrix}F & F^2 & F^2 - wF\end{bmatrix} & \text{if} \quad w < F \\ \begin{bmatrix}0 & 0 & 0\end{bmatrix} \text{ OR } \begin{bmatrix}F & F^2 & 0\end{bmatrix} & \text{if} \quad w = F \\ \begin{bmatrix}0 & 0 & 0\end{bmatrix} & \text{if} \quad w > F \end{cases} \tag{64}$$



By (64),(63),(62)

Input supply function

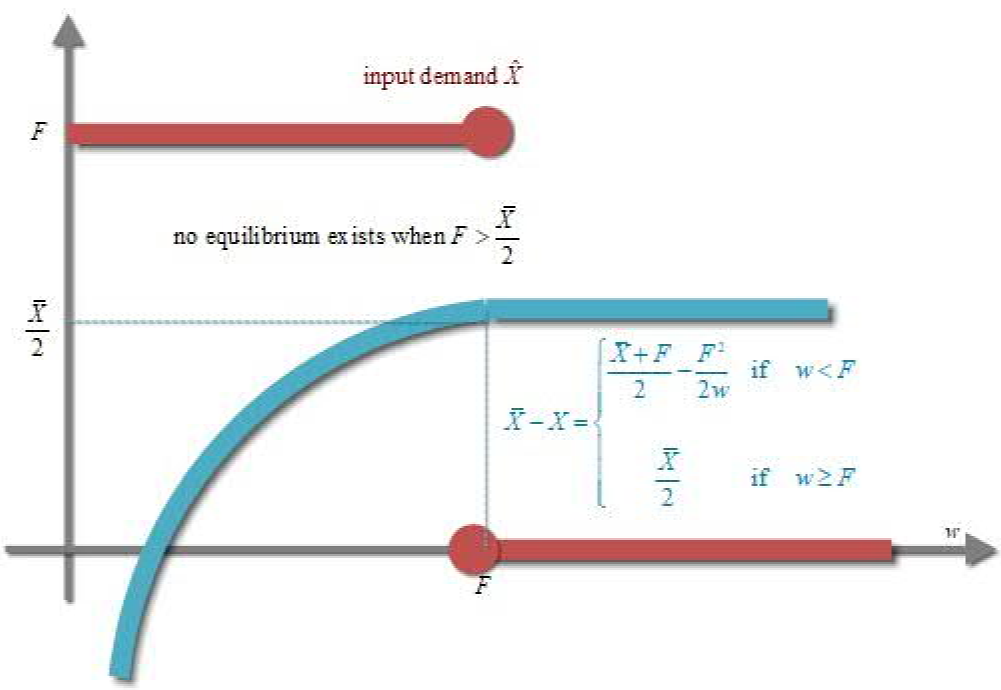
$$\bar{X} - X = \begin{cases} \dfrac{\bar{X}+F}{2} - \dfrac{F^2}{2w} & \text{if} \quad w < F \\ \\ \dfrac{\bar{X}}{2} & \text{if} \quad w \geq F \end{cases} \tag{65}$$

5. SOLVE THE EQUILIBRIUM CONDITIONS

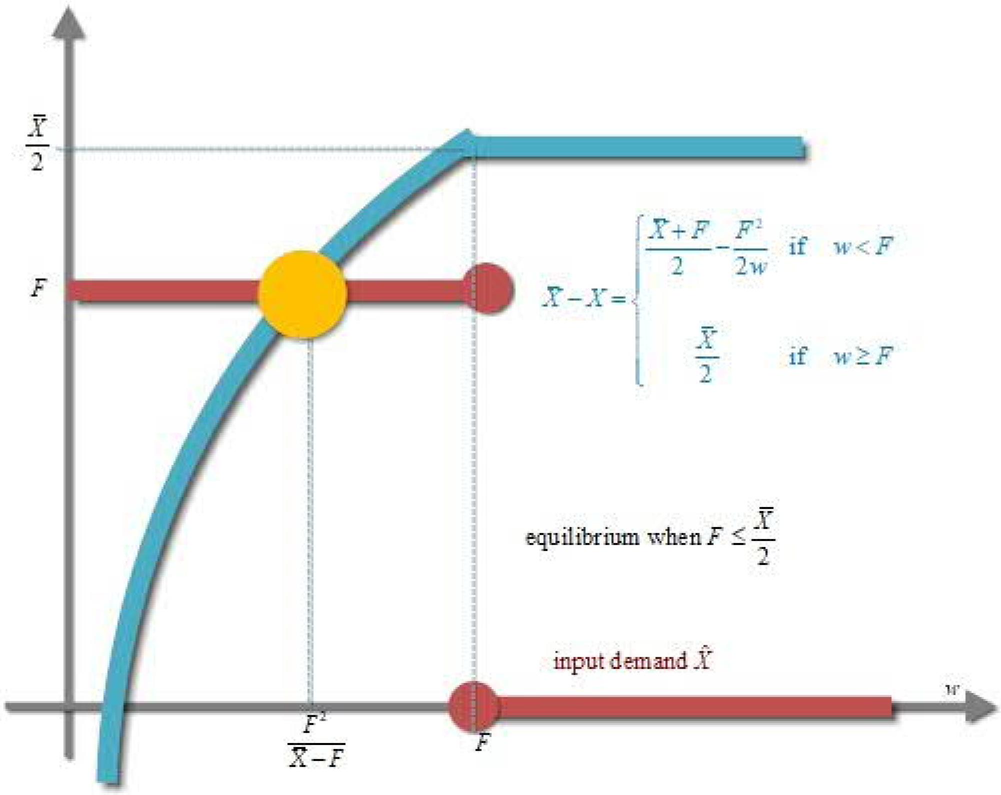
equilibrium conditions

$$A = \hat{A} \tag{66}$$
$$X + \hat{X} = \bar{X}$$

$$\boxed{\begin{array}{l} \text{competitive equilibrium when } \dfrac{\bar{X}}{2} < F \leq \bar{X} \\ \hline \text{no equilibrium exists} \end{array}} \tag{67}$$



$$\boxed{\begin{array}{l}\text{competitive equilibrium when } F \le \dfrac{\overline{X}}{2} \\ w = \dfrac{F^2}{\overline{X}-F}, A = \hat{A} = F^2, X = \overline{X} - F, \hat{X} = F\end{array}} \tag{68}$$



$$\boxed{\begin{array}{l}\text{set of equilibrium allocations} \\ E = \begin{cases} \varnothing & \text{if } \quad F > \dfrac{\overline{X}}{2} \\ \left\{\left[A = \hat{A} = F^2 \quad X = \overline{X} - F \quad \hat{X} = F\right]\right\} & \text{if } \quad F \le \dfrac{\overline{X}}{2}\end{cases}\end{array}} \tag{69}$$

2.EFFICIENT ALLOCATIONS

Since there is only one consumer, both scalarization methods amount to solving the following maximization problem

$$
\begin{aligned}
&\max u = \log A + \log X \text{, subject to} \\
&A \le \hat{A}, X + \hat{X} \le \overline{X}, \\
&\hat{A} = \begin{cases} \hat{X}^2 & \text{if } \quad 0 \le \hat{X} \le F \\ F^2 & \text{if } \quad \hat{X} \ge F \end{cases} \\
&\text{all variables } \ge 0 \\
&\text{variables:} \hat{A}, A, X, \hat{X} \\
&\text{parameters:} F, \overline{X} \\
&\text{conditions on parameters:} 0 \le F < \overline{X}
\end{aligned}
\tag{70}
$$

We transform (70) into the equivalent maximization problem given by

$$
\begin{aligned}
&\max u = 2\log \hat{X} + \log X \text{, subject to} \\
&X + \hat{X} \le \overline{X}, \hat{X} \le F \\
&\text{all variables } \ge 0 \\
&\text{variables:} X, \hat{X} \\
&\text{parameters:} F, \overline{X} \\
&\text{conditions on parameters:} 0 \le F < \overline{X}
\end{aligned}
\tag{71}
$$

Solving (71) we obtain

$$
\boxed{
\begin{array}{l}
\underline{\text{efficient allocations}} \\
P = \begin{cases} \left\{ \left[A = \hat{A} = F^2 \quad X = \overline{X} - F \quad \hat{X} = F \right] \right\} & \text{if} \quad F \le \dfrac{2\overline{X}}{3} \\ \left\{ \left[A = \hat{A} = \left(\dfrac{2\overline{X}}{3} \right)^2 \quad X = \dfrac{\overline{X}}{3} \quad \hat{X} = \dfrac{2\overline{X}}{3} \right] \right\} & \text{if} \quad F \ge \dfrac{2\overline{X}}{3} \end{cases}
\end{array}
}
\tag{72}
$$

3.COMPARISON OF EFFICIENT TO EQUILIBRIUM ALLOCATIONS

By (72),(69) we conclude that

1.the conclusion of the first welfare theorem, namely $E \subseteq P$, holds.

2.the conclusion of the second welfare theorem, namely $P = E$,holds, provided that $E \neq \varnothing$

decentralizable efficient allocations

$$E \bigcap P = \begin{cases} \varnothing & \text{if} \quad F > \dfrac{\bar{X}}{2} \\ E = P & \text{if} \quad F \leq \dfrac{\bar{X}}{2} \end{cases} \tag{73}$$