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Question 1

Consider the function $R^2_{++} \xrightarrow{f} R,\ f(p_1, p_2) = p_1 + p_2$

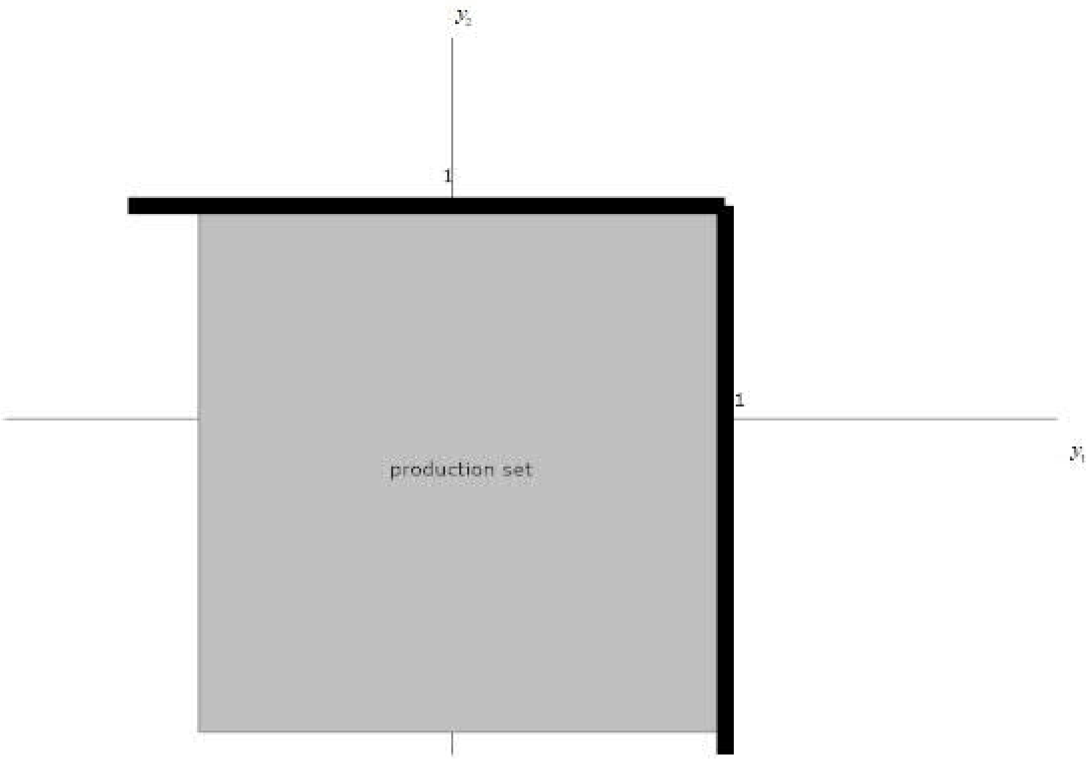
1.is it a profit function?

2.if it is, construct a rationalizing production set for f, and derive the corresponding net supply function

Answer

1.yes.f is convex and positive homogeneous of degree 1

2.$Y = \{y \in \mathbb{R}^2 : py \le f(p) \forall p \gg 0\} = \{y \in \mathbb{R}^2 : y_1 \le 1, y_2 \le 1\}$



The profit maximization problem is $\max\{py : y_1 \le 1, y_2 \le 1\}$,and the net supply function (its solution) is $y_1 = 1, y_2 = 1$.

Question 2

Consider the following price-quantity data (two commodities, three observations)

$$\text{quantity vectors}=\begin{bmatrix} -5 & -5/2 & -1 \\ 3 & 2 & 1 \end{bmatrix} \tag{1}$$

$$\text{price vectors}=\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix} \tag{2}$$

1.show that the data satisfy WAPM

2.Construct and draw the rationalizing production sets YI, YO.

3.Find the profit maximizing output vectors of the firms YI, YO at $p = [6,12]$

4. Find the profit functions of the firms YI, YO

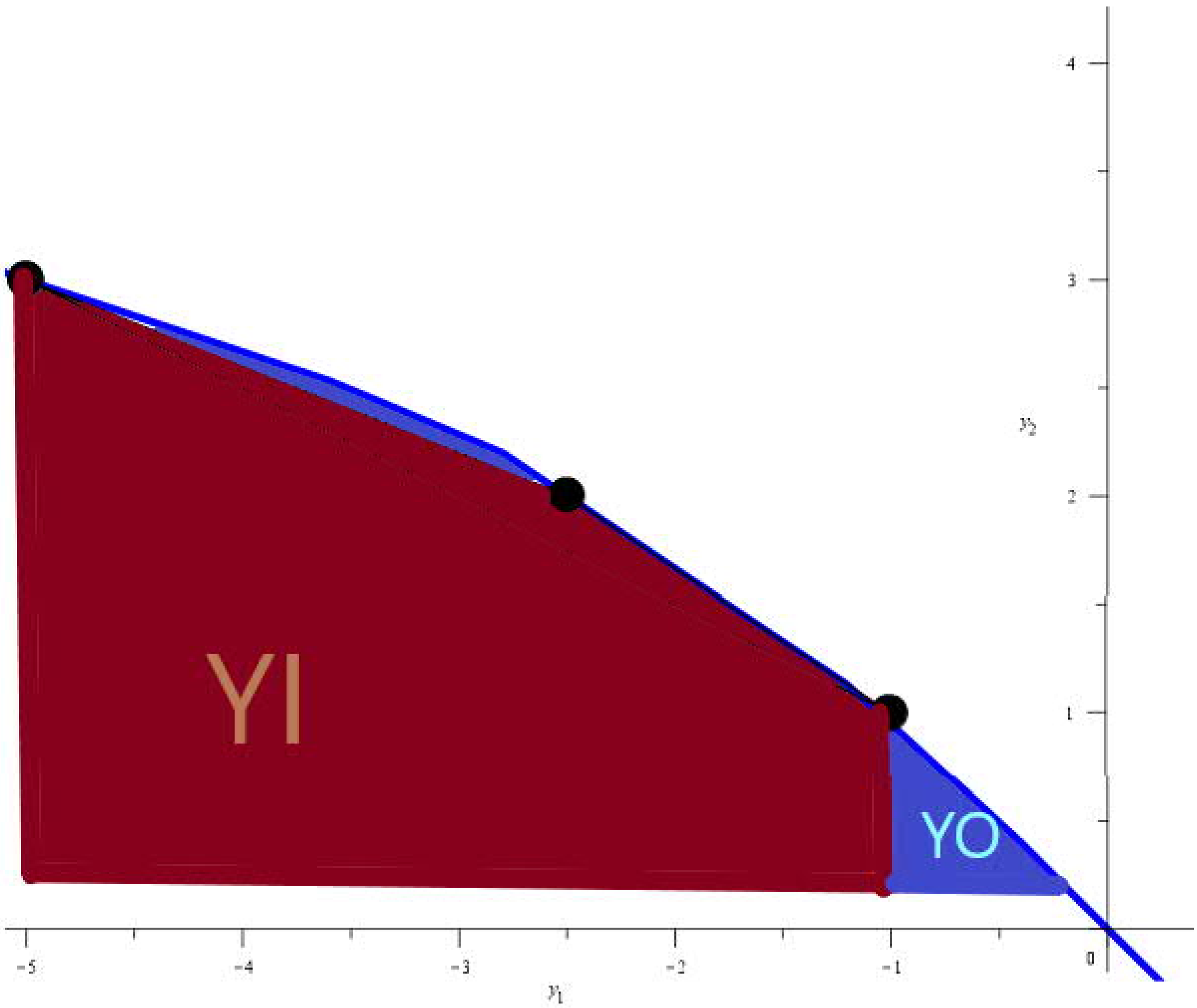
Answer

1.algebra

2.production sets

$$YI = \left\{ \begin{bmatrix} -4t_1 - \frac{3}{2}t_2 - 1 - x_1 \\ 2t_1 + t_2 + 1 - x_2 \end{bmatrix} : t_1 \geq 0, t_2 \geq 0, t_1 + t_2 \leq 1, x_1 \geq 0, x_2 \geq 0 \right\} \tag{3}$$

$$YO = \left\{ y \in \mathbb{R}^2 : 0 \leq 4 - y_1 - 3y_2, 0 \leq 1 - 2y_1 - 3y_2, 0 \leq -3y_1 - 3y_2 \right\} \tag{4}$$



3 Let $p = [6 \quad 12]$.

By (3) the problem $\max\{py : y \in YI\}$ reduces to

$$
\begin{aligned}
&\max(6(-4t_1 - \frac{3}{2}t_2 - 1 - x_1) + 12(2t_1 + t_2 + 1 - x_2)) \\
&t_1 \geq 0, t_2 \geq 0, t_1 + t_2 \leq 1, x_1 \geq 0, x_2 \geq 0 \\
&\text{variables:} t_1, t_2, x_1, x_2
\end{aligned}
\tag{5}
$$

By (4) the problem $\max\{py : y \in YO\}$ reduces to

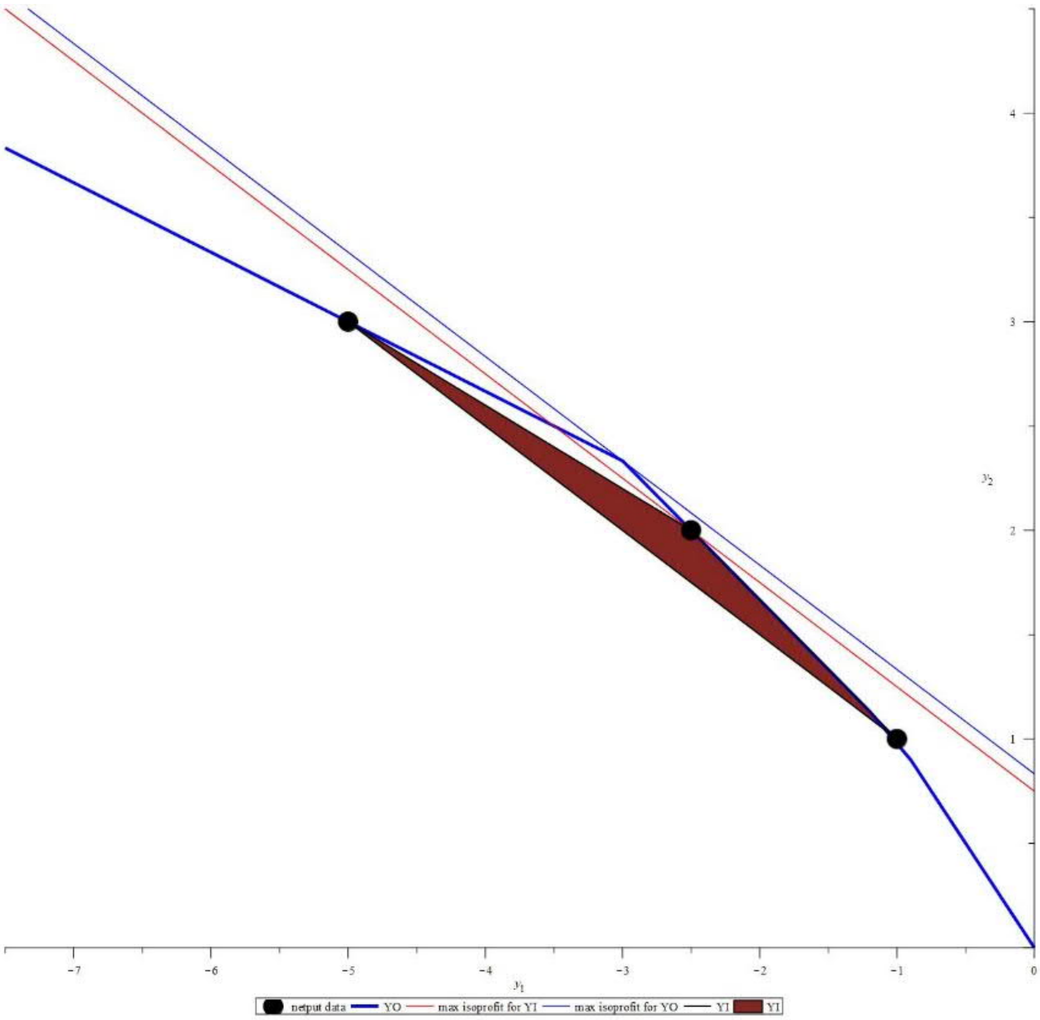
$$
\begin{aligned}
&\max(6y_1 + 12y_2) \\
&0 \leq 4 - y_1 - 3y_2, 0 \leq 1 - 2y_1 - 3y_2, 0 \leq -3y_1 - 3y_2
\end{aligned}
\tag{6}
$$

The solutions of (5) is

$$
y = \begin{bmatrix} -\frac{5}{2} \\ 2 \end{bmatrix}, \text{profit=9} \tag{7}
$$

The solution of (6) is

$$
y = \begin{bmatrix} -3 \\ \frac{7}{3} \end{bmatrix}, \text{profit=10} \tag{8}
$$



4.The profit function $\pi_{YI}\left(p\right)=\max(py,y\in YI)$ is obtained from solving, by (3),

$$\begin{aligned}&\max(p_1(-4t_1-\frac{3}{2}t_2-1-x_1)+p_2(2t_1+t_2+1-x_2))\\&t_1\geq 0,t_2\geq 0,t_1+t_2\leq 1,x_1\geq 0,x_2\geq 0\\&\text{variables:}t_1,t_2,x_1,x_2\end{aligned} \tag{9}$$

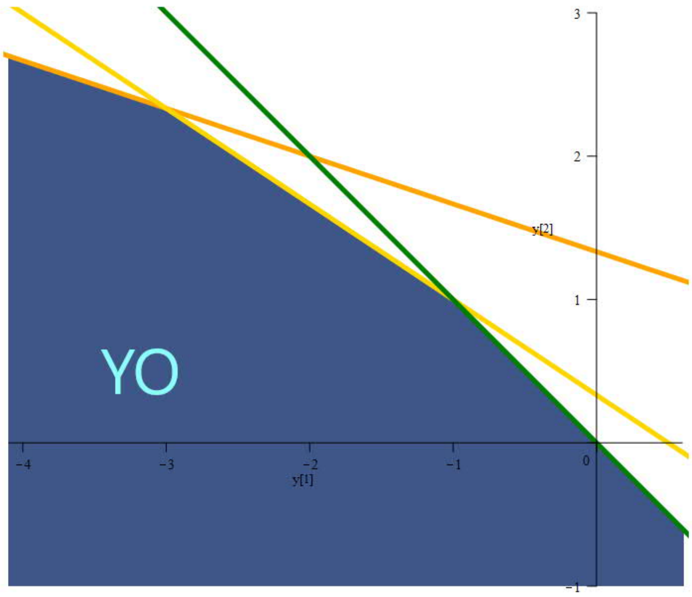
In any solution of (9),we have $x_1=x_2=0$ and

$$\pi_{YI}\left(p\right)=\begin{cases}-p_1+p_2 & p_2\leq\frac{3p_1}{2}\\-\frac{5p_1}{2}+2p_2 & p_2\leq\frac{5p_1}{2}\\-5p_1+3p_2 & \frac{5p_1}{2}<p_2\end{cases} \tag{10}$$

The profit function $\pi_{YO}\left(p\right)=\max(py,y\in YO)$ is obtained from solving, by (4),

$$\begin{aligned}&\max(p_1y_1+p_2y_2)\\&0\leq 4-y_1-3y_2,0\leq 1-2y_1-3y_2,0\leq -3y_1-3y_2\\&\text{variables:}y_1,y_2\end{aligned} \tag{11}$$

$$\pi_{YO}\left(p\right)=\begin{cases}p_2-p_1 & \text{if} \quad p_2\leq\frac{3p_1}{2}\\-3p_1+\frac{7p_2}{3} & \text{if} \quad \frac{3p_1}{2}\leq p_2\leq 3p_1\\\infty & \text{otherwise}\end{cases} \tag{12}$$



Question 3: necessary and sufficient conditions for the existence of a positive representative consumer

Consider two consumers A, B with preferences represented by the utility functions $R_+^L \xrightarrow{U_A} R, R_+^L \xrightarrow{U_B} R$,respectively. Let the corresponding continuous demand functions be $R_{++}^L \times R_+ \xrightarrow{D_A} R_+^L, R_{++}^L \times R_+ \xrightarrow{D_B} R_+^L$ .Define a positive representative consumer C to be a utility function $R_+^L \xrightarrow{U_C} R$ such that the corresponding demand function $R_{++}^L \times R_+ \xrightarrow{D_C} R_+^L$ satisfies $,\forall p \gg 0, \forall m_A \geq 0, \forall m_B \geq 0$

$$\boxed{D_C(p, m_A + m_B) = D_A(p, m_A) + D_B(p, m_B)} \quad (13)$$

Find conditions on the continuous demand functions (D_A, D_B) that are necessary and sufficient for a positive representative consumer to exist.

Hint1:do not differentiate the demand functions. Hint2: Cauchy's functional equation

Answer

The necessary and sufficient conditions are

1.demand functions are identical

$$D_A(p, w) = D_B(p, w) = D_C(p, w), \forall p \gg 0, \forall w \geq 0 \quad (14)$$

And

2.Demand functions are linear in income (linear Engel curves)

$$D_C(p, w) = D_C(p, 1)w, \forall p \gg 0, \forall w \geq 0 \quad (15)$$

Proof of sufficiency

We have to show that (14) and (15) imply (13).

$$\begin{aligned} & D_A(p, m_A) + D_B(p, m_B) \\ & = m_A D_C(p, 1) + m_B D_C(p, 1) \\ & = (m_A + m_B) D_C(p, 1) \\ & = D_C(p, m_A + m_B) \end{aligned}$$

Proof of necessity

We have to show that (13) implies (14) and (15).

$$\begin{aligned} D_C(p, w) = D_C(p, w + 0) = D_A(p, w) + D_B(p, 0) = D_A(p, w) + 0 = D_A(p, w) \\ D_C(p, w) = D_C(p, 0 + w) = D_A(p, 0) + D_B(p, w) = 0 + D_B(p, w) = D_B(p, w) \end{aligned}$$

Hence (14) holds.

To show (15),fix the price vector p and define $f(w) = D_C(p, w)$ .Then (13) implies that f satisfies Cauchy's integral equation

$$f(x+y)=f(x)+f(y),\forall x,\forall y$$

Since f is continuous, it has to be linear, i.e. $f(x)=xf(1),\forall x$ .Hence

$$D_C(p,w)=f(w)=wf(1)=wD_C(p,1),\forall w,\forall p$$

Question 4: positive vs. normative representative consumer

Consider two consumers A, B with utility functions and endowment vectors given by

$$\begin{aligned} &U_A(A_1,A_2,B_1,B_2)=\log A_1+\log A_2-4B_1 \\ &\omega_A=[1,1] \\ &U_B(A_1,A_2,B_1,B_2)=\log B_1+\log B_2-4A_1 \\ &\omega_B=[1,1] \end{aligned} \tag{16}$$

Consumer A chooses the vector (A_1,A_2),and consumer B chooses the vector (B_1,B_2).

1.Compute the demand functions $R^2_{++}\xrightarrow{D_A}R^2_+, R^2_{++}\xrightarrow{D_B}R^2_+$,and the indirect utility functions $v_A(p)=U_A(D_A(p)),v_B(p)=U_B(D_B(p))$ of consumers A, B respectively

2.Compute the demand function $R^2_{++}\xrightarrow{D_C}R^2_+$,and the indirect utility function $v_C(p)=U_C(D_C(p))$ of a consumer C with utility function and endowment vector given by

$$\begin{aligned} &U_C(C_1,C_2)=\log C_1+\log C_2 \\ &\omega_C=[2,2] \end{aligned} \tag{17}$$

3.Show that consumer C is a positive representative consumer in the sense that for all $p\gg 0$,

$$\boxed{D_C(p)=D_A(p)+D_B(p)} \tag{18}$$

4.show that the preferences of the positive representative consumer C have no normative significance, in the sense that there exist price vectors p,p' such that both consumers A,B strictly prefer p to p',i.e. $v_A(p)>v_A(p'),v_B(p)>v_B(p')$, while the representative consumer C strictly prefers p' to p,i.e. $v_C(p')>v_C(p)$.

Answer

$$D_A(p)=\begin{bmatrix}A_1\\A_2\end{bmatrix}=\begin{bmatrix}\frac{p_1+p_2}{2p_1}\\ \frac{p_1+p_2}{2p_2}\end{bmatrix}=\begin{bmatrix}B_1\\B_2\end{bmatrix}=D_B(p) \tag{19}$$

$$v_A(p)=v_B(p)=2\log(p_1+p_2)-\log p_1-\log p_2-2\frac{p_1+p_2}{p_1}-2\log 2 \tag{20}$$

$$D_C(p) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \frac{p_1 + p_2}{p_1} \\ \frac{p_1 + p_2}{p_2} \end{bmatrix} \tag{21}$$

$$v_C(p) = 2\log(p_1 + p_2) - \log p_1 - \log p_2 \tag{22}$$

Equations (19) and (21) imply (18),hence C is a positive representative consumer.

To see that C is not a representative normative consumer, differentiate $v_A = v_B$ and v_C w.r.t p_1 ,and find where the signs of the derivatives are opposite

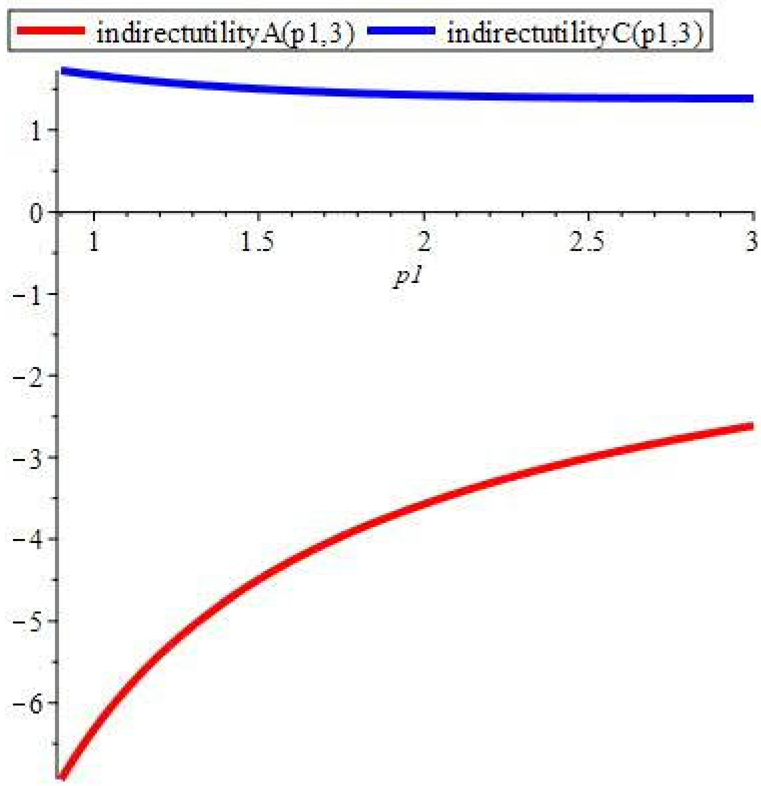
$$\frac{\partial v_A}{\partial p_1} = \frac{\partial v_B}{\partial p_1} = \frac{p_1^2 + p_1 p_2 + 2p_2^2}{(p_1 + p_2)p_1^2} \tag{23}$$

$$\frac{\partial v_C}{\partial p_1} = \frac{p_1 - p_2}{(p_1 + p_2)p_1} \tag{24}$$

The derivatives have opposite signs iff $p_1 < p_2$. For example, $\frac{\partial v_A(1,3)}{\partial p_1} = \frac{11}{2}, \frac{\partial v_C(1,3)}{\partial p_1} = -\frac{1}{2}$

Let $p = [1,3], p' = [2,3]$ .Then

$$v_A(p) = \text{-}6.326 < -3.572 = v_A(p')$$
$$v_C(p) = 1.673 > 1.427 = v_C(p')$$



Question 5: GARP and the representative consumer

Let A, B be two consumers about whom we have the datasets $F_A = [(p^1, A^1), (p^2, A^2)], F_B = [(p^1, B^1), (p^2, B^2)]$,where

$$\begin{aligned} p^1 &= [3, 2, 1], p^2 = [2, 3, 2] \\ A^1 &= [3, 0, 0], A^2 = [2, 0, 0] \\ B^1 &= [0, 1, 0], B^2 = [0, 2, \theta], \theta \geq 0 \end{aligned} \tag{25}$$

Let $F = [(p^1, A^1 + B^1), (p^2, A^2 + B^2)]$

1.Compute the set

$$\Theta_{A,B} = \{\theta \geq 0, \text{the datasets } F_A, F_B \text{ both satisfy GARP}\} \tag{26}$$

2. Compute the set

$$\Theta_{A+B} = \{\theta \geq 0, \text{the dataset } F \text{ fails GARP}\} \tag{27}$$

3.Suppose $\theta \in \Theta_{A,B} \cap \Theta_{A+B}$ .Does there exist a positive representative consumer?

Answer

$$\text{Afriat matrix}(F_A) = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \tag{28}$$

By (28) there are no negative cycles in the revealed preference graph of consumer A, hence A satisfies GARP.

$$\text{Afriat matrix}(F_B) = \begin{bmatrix} 0 & 2+\theta \\ -3-2\theta & 0 \end{bmatrix} \tag{29}$$

By (29) there are no negative cycles in the revealed preference graph of consumer B, because if one of the off-diagonal elements of its afriat matrix is negative, the other is necessarily positive; hence B satisfies GARP. Hence

$$\Theta_{A,B} = \{\theta \geq 0, \text{the datasets } F_A, F_B \text{ both satisfy GARP}\} = \mathbb{R}_+ \tag{30}$$

$$\text{Afriat matrix}(F) = \begin{bmatrix} 0 & -1+\theta \\ -1-2\theta & 0 \end{bmatrix} \tag{31}$$

By (31),the dataset F violates GARP if and only if $0 \leq \theta \leq 1$,i.e.

$$\Theta_{A+B} = \{\theta \geq 0, \text{the dataset } F \text{ fails GARP}\} = [0,1] \tag{32}$$

Hence there is no representative consumer if $0 \leq \theta \leq 1$.

Question 6: profit taxation

the economy

- N+1 consumers.
- Two goods: A and X.
- One firm, with production function: $\hat{A} = \sqrt{2\hat{X}}$
- Consumer 0 has no endowment of either good. He is the sole owner of the firm. His preferences are given by the utility function $U_0 = X_0$.
- Consumers i=1...N have preferences described by the utility functions $U_i = \log A_i + \log X_i$ Each owns one unit of good X.
- Profits Π are taxed at a rate $0 \le t < 1$. Tax revenue $R = t\Pi$ is returned to consumers 1,2..,N with lump sum transfers $T_i = R / N$.

1. Compute competitive equilibria as a function of the policy parameter t

2. Plot the equilibrium values of all variables as a function of the tax rate

1. NAME THE PRICE OF EACH GOOD

$p = \text{price of } A, w = \text{price of } x$

3. SOLVE the optimization problems of firms

Post-tax profit $F = (1-t)\Pi = (1-t)(p\hat{A} - w\hat{X}) = (1-t)(p\hat{A} - w\frac{\hat{A}^2}{2})$ is maximized at

$$\begin{bmatrix} \hat{A} \\ \hat{X} \\ \Pi \end{bmatrix} = \begin{bmatrix} p/w \\ (p/w)^2/2 \\ p^2/2w \end{bmatrix} \tag{33}$$

4. SOLVE the optimization problems of consumers

$\max U_0 = X_0, \text{subject to } wX_0 \le (1-t)\Pi$

$\max U_i = \log X_i + \log A_i, \text{subject to } wX_i + pA_i \le w + t\frac{\Pi}{N}, i = 1..N$

The solutions are

$$\begin{bmatrix} X_0 \\ A_0 \end{bmatrix} = \begin{bmatrix} (1-t)\Pi / w \\ 0 \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} X_i \\ A_i \end{bmatrix} = \begin{bmatrix} M/2w \\ M/2p \end{bmatrix}, M = w + t\frac{\Pi}{N} \tag{35}$$

5. SOLVE the equilibrium conditions

$$\sum_{i=0}^{N} X_i + \hat{X} = N, \sum_{i=0}^{N} A_i = \hat{A} \tag{36}$$

There is a unique solution, described by

EQUILIBRIUM WITH PROFIT TAXATION $0 \le t < 1$

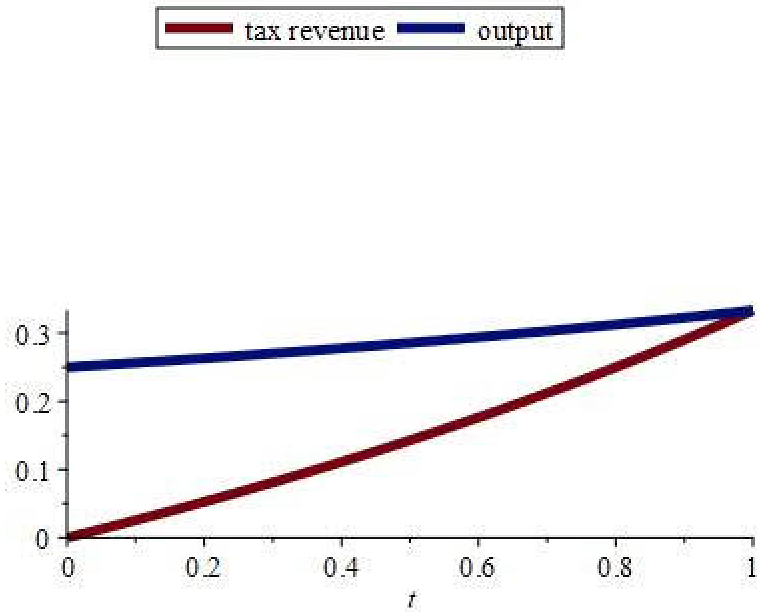
$$\frac{w}{p} = \frac{\sqrt{4-t}}{\sqrt{2N}}, \frac{M}{w} = \frac{4}{4-t}$$

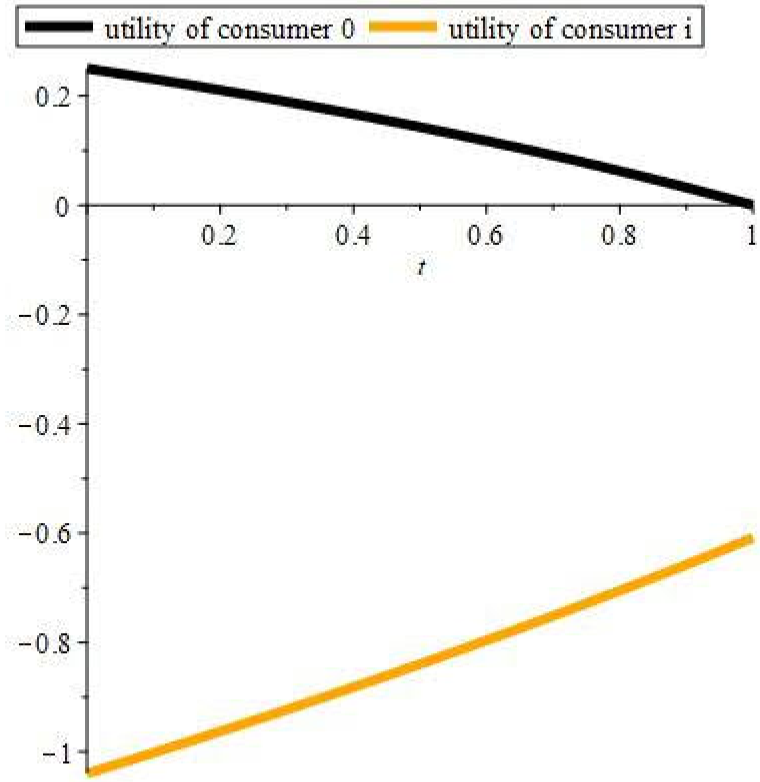
$$\frac{\Pi}{w} = \frac{N}{4-t}, \frac{R}{w} = \frac{tN}{4-t}$$

$$\hat{A} = \frac{\sqrt{N}\sqrt{2}}{\sqrt{4-t}}, \hat{X} = \frac{N}{4-t}$$

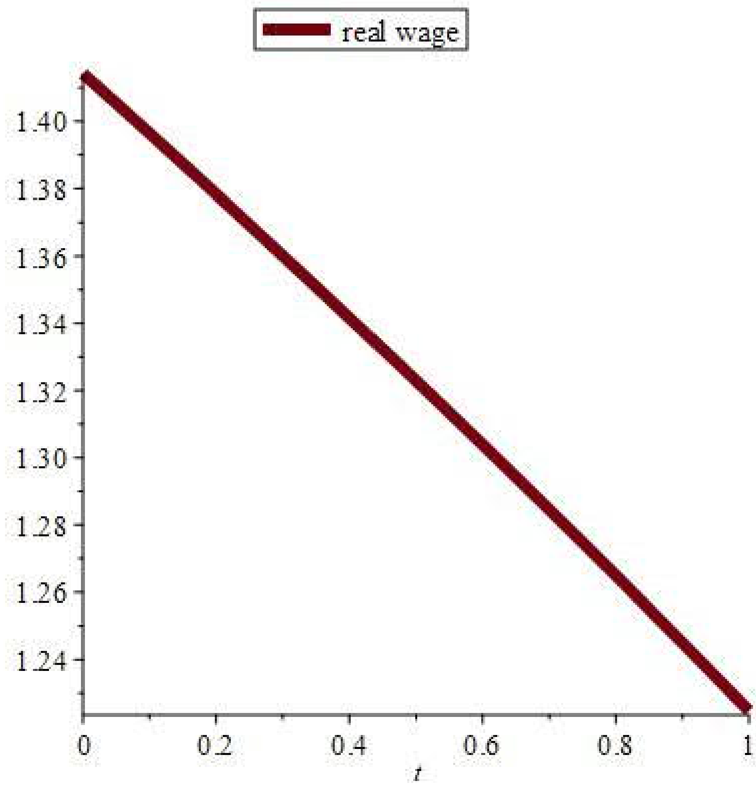
$$X_i = \frac{2}{4-t}, A_i = \frac{\sqrt{2}}{\sqrt{N}\sqrt{4-t}}$$

$$X_0 = \frac{N(1-t)}{4-t}, A_0 = 0$$





utility of consumer 0
utility of consumer i
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0.2
0.4
0.6
0.8
1
t



real wage
1.40
1.38
1.36
1.34
1.32
1.30
1.28
1.26
1.24
0
0.2
0.4
0.6
0.8
1
t

Question 7: commodity taxation

the economy

- Consumers A, B
- Goods 1, 2.
- Preferences

$$u_A = 2\sqrt{A_1 A_2}, u_B = 2\sqrt{B_1 B_2}$$

- Endowments $e_A = [1,0], e_B = [0,1]$
- Consumers pay a tax $-1 < t$ for each unit of good 1 they buy (hence only consumer B pays taxes).
- Tax revenue R is distributed to consumers with lump-sum transfers

  $T_A = \alpha R, T_B = (1-\alpha)R, 0 \le \alpha \le 1.$

1. Compute competitive equilibria as a function of the policy parameters t, α

2. Plot the equilibrium values of all variables as a function of the tax rate

ANSWERS

1.Name the price of each good

$p_i =$ price of good i

2. Define consumer incomes

$$M_A = p_1 + T_A, M_B = p_2 + T_B \tag{37}$$

3. SOLVE the optimization problems of consumers

$\max U_A = \log A_1 + \log A_2$, subject to $p_1 A_1 + p_2 A_2 \le M_A$

$\max U_B = \log B_1 + \log B_2$, subject to $p_1 B_1 + t B_1 + p_2 B_2 \le M_B$

The solutions are

$$(A_1, A_2) = \left(\frac{M_A}{2p_1}, \frac{M_A}{2p_2} \right) \tag{38}$$

$$(B_1, B_2) = \left(\frac{M_B}{2(p_1 + t)}, \frac{M_B}{2p_2} \right) \tag{39}$$

5. SOLVE the equilibrium conditions

$$1 = A_1 + B_1, 1 = A_2 + B_2, T_A + T_B = tB_1 \tag{40}$$

There is a unique solution, expressed in terms of the parameter $\tau = \frac{t}{p_1}$

competitive equilibria with unit taxation

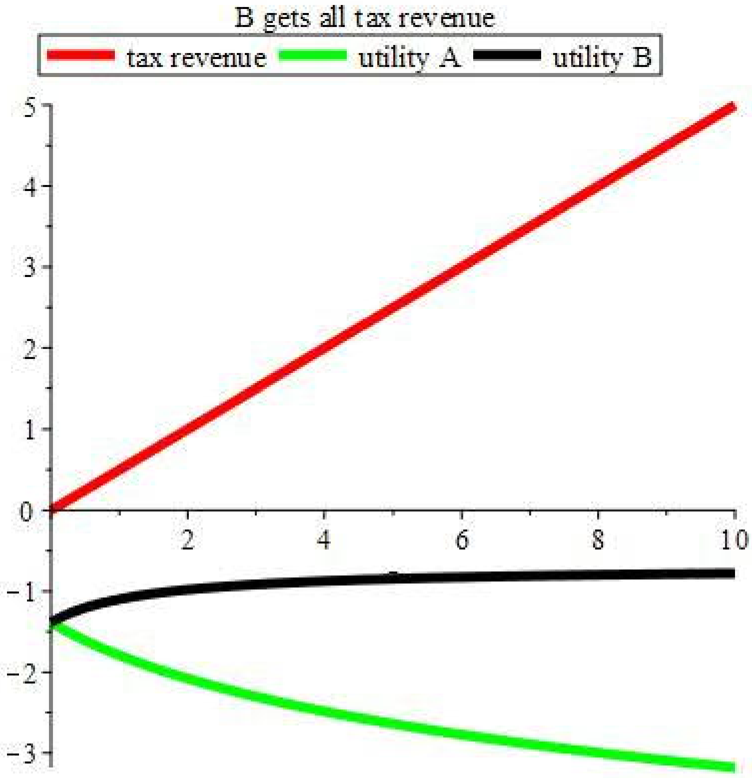
$$\frac{p_2}{p_1} = \frac{\alpha\tau + \tau + 2}{\alpha\tau + 2}$$

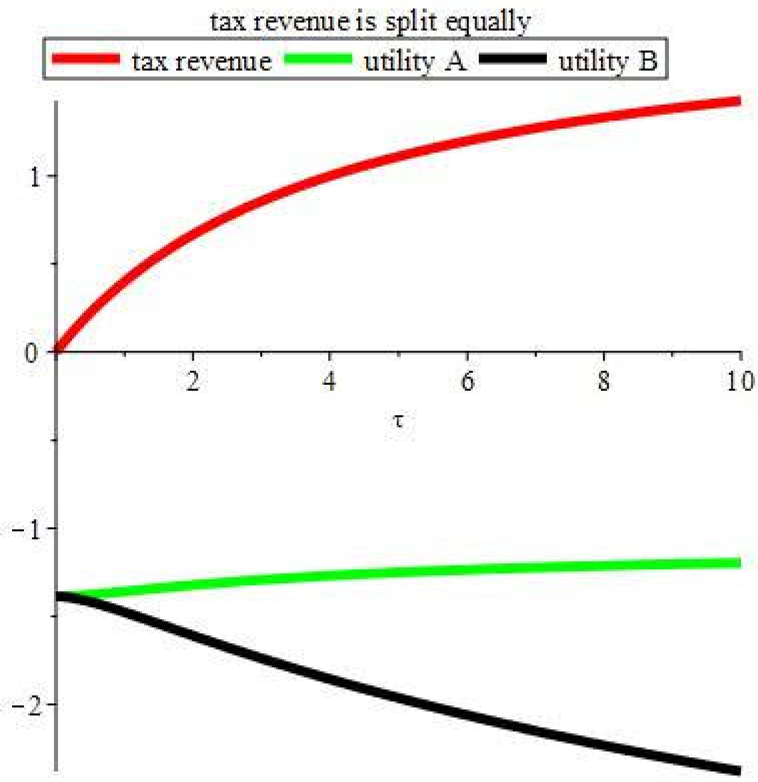
$$\frac{T_A}{p_1} = \frac{\alpha\tau}{\alpha\tau + 2}, \frac{T_B}{p_1} = \frac{\tau(1-\alpha)}{\alpha\tau + 2}$$

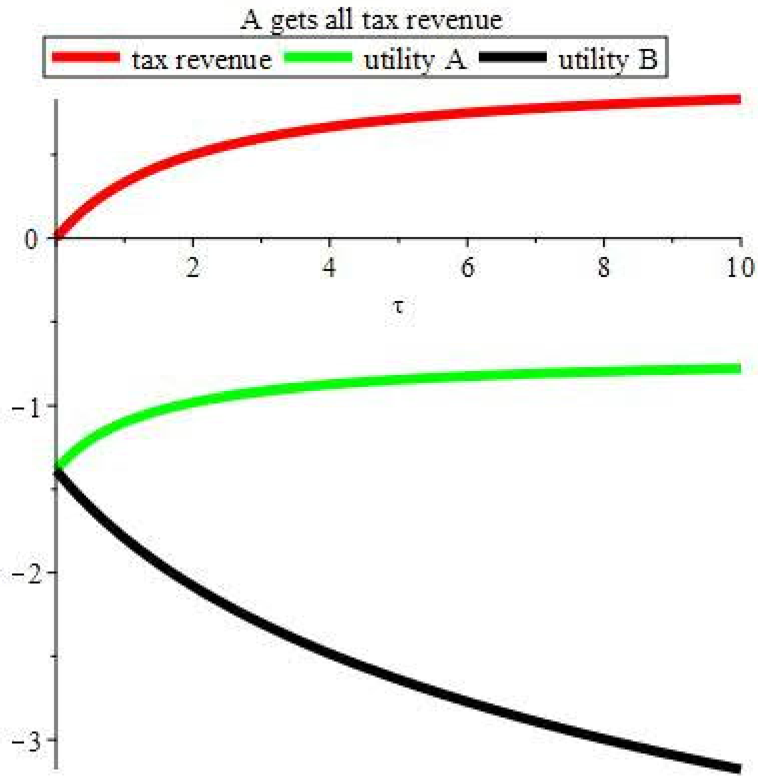
$$A = \left[\frac{\alpha\tau + 1}{\alpha\tau + 2}, \frac{\alpha\tau + 1}{\alpha\tau + \tau + 2}\right] \tag{41}$$

$$B = \left[\frac{1}{\alpha\tau + 2}, \frac{1 + \tau}{\alpha\tau + \tau + 2}\right]$$

$$R = \frac{\tau}{\alpha\tau + 2}$$




tax revenue is split equally
tax revenue
utility A
utility B
1
0
-1
-2
2
4
6
8
10
τ


A gets all tax revenue
tax revenue
utility A
utility B
0
-1
-2
-3
2
4
6
8
10
τ