## THE ECONOMY

- N consumers
- Two goods. Good A is public, good x is private
- Preferences are described by $u_i = \log x_i + \theta \log A, \theta > 0$
- Each consumer has one unit of the private good,and an equal share in firm profits.
- The public good is produced out of the private good by a single firm with technology described by the production function $\hat{A} = X^k, 0 < k < 1$

## QUESTIONS

1.Compute all pareto efficient allocations.

2. Suppose that the firm is paid a subsidy $s > 0$ per unit of output,financed with a lump-sum tax T on each consumer. Compute the level of the subsidy at which the corresponding competitive equilibrium is pareto efficient.

3.Suppose that the firm is the only one to know the true value of k.Is the equilibrium allocation that you computed in part 2 implementable?If not,does the firm underreport or overreport the value of its productivity parameter k?

# ANSWERS

## EFFICIENT POINTS

Efficient points are the solutions of

$$\max \sum_{i=1}^{N} \alpha_i u_i = \sum_{i=1}^{N} \alpha_i \log x_i + \theta k \log X, \text{subject to X+} \sum_{i=1}^{N} x_i \leq N \text{ ,ie}$$

$$\boxed{\begin{aligned} & x_i^{pareto} = \frac{\alpha_i N}{1+\theta k}, X^{pareto} = \frac{\theta k N}{1+\theta k} \\ & u_i^{pareto} = \log(\frac{\alpha_i N}{1+\theta k}) + \theta \log\left(\frac{\theta k N}{1+\theta k}\right) \end{aligned}} \quad (1)$$

## COMPETITIVE EQUILIBRIUM WITH SUBSIDIES

Let $s$ be a subsidy paid to the firm per unit of output. The subsidy is financed by a lump-sum tax T on each consumer, equal for all consumers. We will compute competitive equilibria under such a tax-subsidy scheme,only for the value of the subsidy that equates the equilibrium value of the public good to its pareto optimal value.

## 1. NAME THE PRICE OF EACH GOOD

$p = \text{price of } A, w = \text{price of } x$

## 2. NORMALIZE PRICES (OPTIONAL)

$w = 1$

## 3. SOLVE THE OPTIMIZATION PROBLEMS OF FIRMS

$\Pi = p\hat{A} - wX + s\hat{A} = (p+s)X^k - X$ *is maximized at*

$$\begin{bmatrix} \hat{A} \\ X \\ \Pi \end{bmatrix} = \begin{bmatrix} k^{\frac{k}{1-k}}(p+s)^{\frac{k}{1-k}} \\ k^{\frac{1}{1-k}}(p+s)^{\frac{1}{1-k}} \\ (1-k)k^{\frac{k}{1-k}}(p+s)^{\frac{1}{1-k}} \end{bmatrix} \tag{2}$$

## 4. SOLVE THE OPTIMIZATION PROBLEMS OF CONSUMERS

$\max U_i = \log x_i + \theta \log A, \text{subject to } x_i + pA_i \leq 1 - T + \frac{\Pi}{N}, A = A_i + \sum_{j \neq i}^{N} A_j$ The solution is

$$(x_i, A_i) = \begin{cases} \left( \dfrac{m + pA_{-i}}{1+\theta}, \dfrac{\theta m - pA_{-i}}{p(1+\theta)} \right) & \text{if } \quad A_{-i} \leq \dfrac{\theta m}{p} \\ (m, 0) & \text{if } \quad A_{-i} \geq \dfrac{\theta m}{p} \end{cases} \tag{3}$$

$$m = 1 - T + \frac{\Pi}{N}, A_{-i} = \sum_{j \neq i}^{N} A_j$$

## 5. SOLVE THE EQUILIBRIUM CONDITIONS

$$\sum_{i=1}^{N} x_i + X = N, \sum_{i=1}^{N} A_i = \hat{A} = X^k, s\hat{A} = NT, X = X^{pareto}, \hat{A} = A^{pareto} \tag{4}$$

There is a unique symmetric solution, described by

$$
\boxed{\begin{array}{l}
p = N^{(-k)}\theta^{(1-k)}k^{(-k)}(1+\theta k)^{(k-1)} \\
s = (N-1)N^{(-k)}\theta^{(1-k)}k^{(-k)}(1+\theta k)^{(k-1)} \\
T = \dfrac{\theta(N-1)}{(1+\theta k)N}, m = \dfrac{N+\theta}{(1+\theta k)N} \\
x_i = \dfrac{1}{1+\theta k} \\
X = \dfrac{\theta k N}{1+\theta k}, A = \left(\dfrac{\theta k N}{1+\theta k}\right)^k
\end{array}} \qquad (5)
$$

## 6.IMPLEMENTABILITY OF MECHANISM (5)

The profit function of the firm under mechanism (5) becomes

$$
\begin{aligned}
\Pi(k,\rho) &= (p(k)+s(k))X(k)^{\rho} - X(k) \\
&= N\theta \frac{[N^{\rho-k}k^{\rho-k}\theta^{\rho-k}(1+\theta k)^{k-\rho} - k]}{(1+\theta k)}
\end{aligned} \qquad (6)
$$

where k is the declared value,and $\rho$ the true value,of the productivity parameter.

A necessary condition for truth-telling for all values of k is

$$
\left.\frac{\partial \Pi(k,\rho)}{\partial k}\right|_{k=\rho} = 0, \forall \rho \in (0,1) \qquad (7)
$$

By (5),(6) we have

$$
\left.\frac{\partial \Pi(k,\rho)}{\partial k}\right|_{k=\rho} = \frac{N\theta}{(1+\theta\rho)^2}\left[(1+\theta\rho)\log\left(\frac{1+\theta\rho}{\theta\rho N}\right) - 1 - \theta\right] \qquad (8)
$$

The expression (8) is not identically zero ,hence (5) is not implementable.For $N=2, \theta=1$ for example, the graph of (8) is



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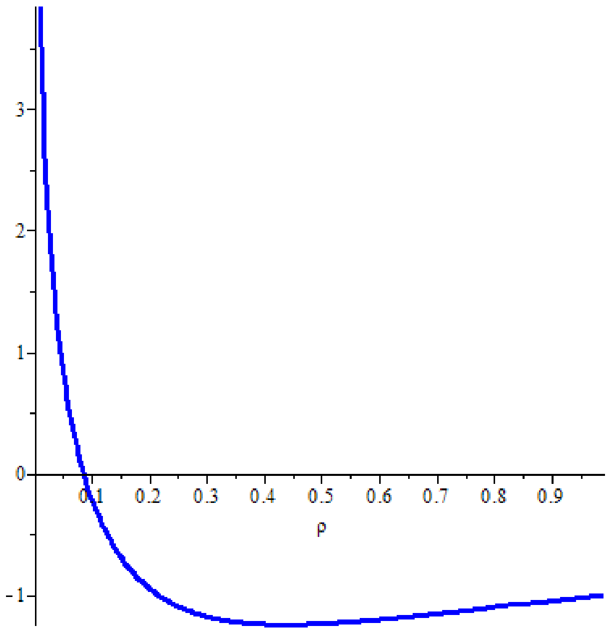


showing that the firm has an incentive to overreport low values of the productivity parameter,and to underreport high values of the productivity parameter.