Financial Mathematics

Portfolios

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A portfolio is a collection of assets of the same or different type used for investment purposes (speculation or risk management)

Portfolios may consist of stocks, bonds and derivative assets

The aim of portfolio theory is to provide quantitative tools as well as techniques and strategies of constructing the appropriate portfolio for the appropriate task:

- Risk
- Return

Returns

Suppose you have an asset i whose price in the market changes between times t and $t+1$ from $P_i(t)$ to $P_i(t+1)$ respectively.

Its return is the random variable

$$R_i(t) = \frac{P_i(t+1) - P_i(t)}{P_i(t)} \simeq \ln\left(\frac{P_i(t+1)}{P_i(t)}\right)$$

We have seen that returns can be stationary or somes times even uncorrelated in real markets for some periods, whereas for certain periods there may be strong corellations between them (recall e.g. the Black-Schoes vs the Garch model).

Moreover, for some periods the returns can be well represented by the first two moments.

Portfolios

In a real market we have many assets, with corresponding

- price processes

$$\{P_i(t) \ : t \in \mathbb{N}\}, \ \ i = 1, \cdots, N$$

- return processes

$$\{R_i(t) = \frac{P_i(t+1) - P_i(t)}{P_i(t)} \simeq \ln\left(\frac{P_i(t+1)}{P_i(t)}\right) \ : \ t \in \mathbb{N}\},, \ \ i = 1, \cdots, N$$

Definition

A portfolio is a collection of assets in the market, identified by a vector $(\theta_1(t), \cdots, \theta_N(t)) \in \mathbb{R}^N$ with θ_i being the position held in asset i in the period $[t, t+1]$

- $\theta_i \geq 0$ long position
- $\theta_i < 0$ short position

Portfolio returns

The value of the portfolio θ at times t and $t+1$ is respectively

$$V^\theta(t) = \sum_{i=1}^N \theta_i(t) P_i(t),$$
$$V^\theta(t+1) = \sum_{i=1}^N \theta_i(t) P_i(t+1),$$

It return for this period is then

$$R^\theta(t) = \frac{V^\theta(t+1) - V^\theta(t)}{V^\theta(t)} = \sum_{i=1}^N \frac{\theta_i(t)}{V^\theta(t)}(P_i(t+1) - P_i(t))$$
$$= \sum_{i=1}^N \frac{\theta_i(t)P_i(t)}{V^\theta(t)} \frac{(P_i(t+1) - P_i(t))}{P_i(t)} = \sum_{i=1}^N w_i(t) R_i(t),$$

where

$$w_i(t) := \frac{\theta_i(t) P_i(t)}{V^\theta(t)}$$

is the fraction of the portfolio wealth allocated to asset i.

Hence, the portfolio return is the random variable $R^\theta(t) = \sum_{i=1}^N w_i(t) R_i(t)$, i.e. the weighted average of the returns of the assets.

The distribution of $R^\theta(t)$ can determined by the joint distribution of the vector process $(R_1(t), \cdots, R_N(t))$.
A reduced description of this random variable can be given in terms of the first two moments

- Mean return

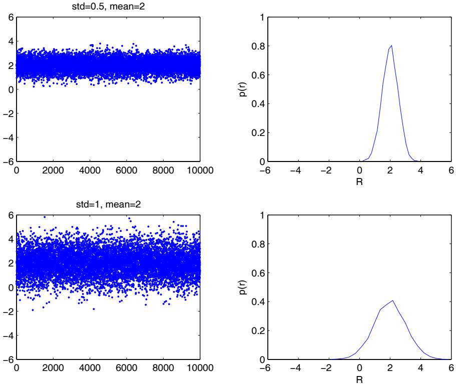
$$\mathbb{E}[R^\theta(t)] = \sum_{i=1}^N w_i(t)\mathbb{E}[R_i(t)]$$

  Best estimate for the realizations of the return

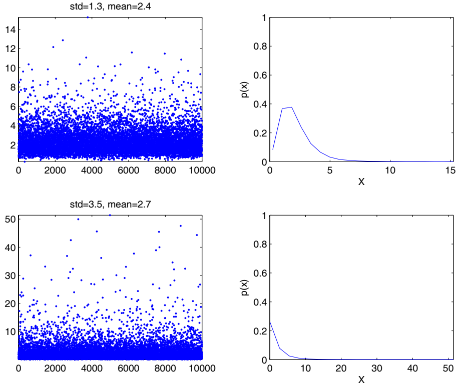
- Variance

$$Var(R^\theta(t)) = \mathbb{E}[(R^\theta(t) - \mathbb{E}[R^\theta(t)])^2] = \sum_{i=1}^N \sum_{j=1}^N w_i(t) w_j(t) Cov(R_i(t) R_j(t))$$

  Risk of the portfolio (recall Chebychev inequality)

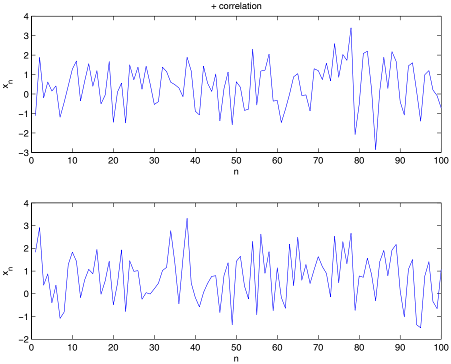


Variance as a risk measure: The normal distribution

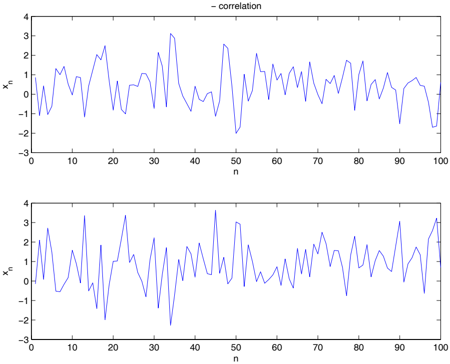


Variance as a risk measure: The lognormal distribution

Covariance and its role



Positive covariance



Negative covariance

Differentiation

The variance of the portfolio can be expressed as

$$Var(R) = \sum_{i=1}^{N} w_i^2 Var(R_i) + \sum_{i=1}^{N} \sum_{j=1, i \neq j}^{N} w_i w_j Cov(R_i, R_j)$$

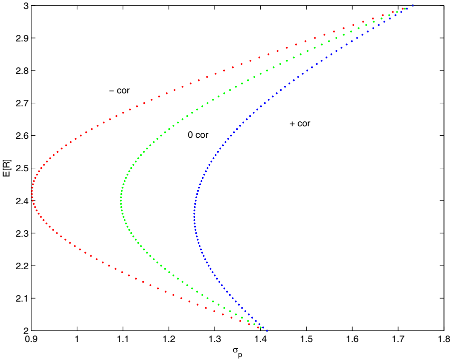
The first part of this sum is always positive.

The second part takes positive or negative values depending on the sign of $Cov(R_i, R_j)$

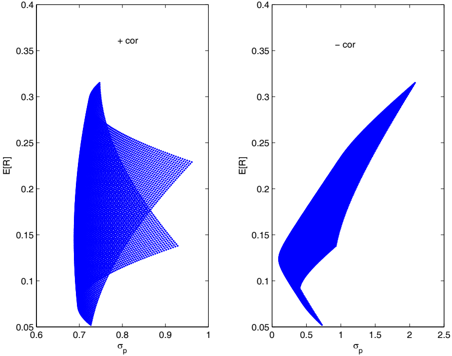
We may decrease the risk as quantified by $Var(R)$ by appropriate choosing w_i, w_j among the various assets i, j that are negatively corellated.

This is called differentiation:

> *... it is generally more likely for firms within the same industry to do poorly at the same time than for firms in dissimilar industries, H. Markowitz*



$\mathbb{E}[R]$-σ_p diagram for portfolios consisting of two assets for different covariance between the assets



$\mathbb{E}[R]$-σ_p diagram for portfolios consisting of three assets for different covariance between the assets

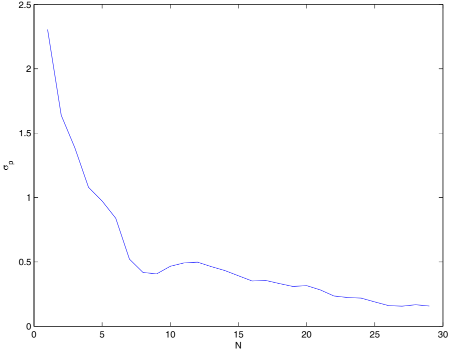
Fully differentiated portfolio $w_i = 1/N$

The variance of the portfolio can be expressed as

$$\begin{aligned}
\sigma_P^2 = Var(P) &= \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} \\
&= \frac{1}{N}\left(\frac{1}{N}\sum_{i=1}\sigma_i^2\right) + \frac{N-1}{N}\left(\frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\sigma_{ij}\right) \\
&= \frac{1}{N}\bar{\sigma}^2 + \frac{N-1}{N}\bar{c} \\
&= \bar{c} + \frac{1}{N}(\bar{\sigma}^2 - \bar{c})
\end{aligned}$$

- $\bar{\sigma}^2$ mean of the variances of the of the returns of the N assets
- $\bar{c}$ is the mean of the $N(N-1)$ covariances of the asset i with the remaining $N-1$ assets

As $N \to \infty$ we have that $\sigma_p^2 \to \bar{c}$ — systematic (non-differentiable) risk.



σ_p for a perfectly differentiated portfolio as a function of N

The Markowitz problem: Minimum risk given return

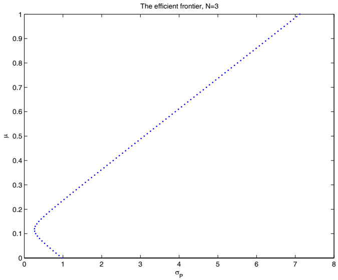
Given $\mathbb{E}[R_i]$,$\sigma_i^2 = \mathbb{E}[(R_i - E[R_i])^2]$, $Cov(R_i, R_j)$ and $\mu \in \mathbb{R}$, solve the problem

$$\min_{\{w_i\}} \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, i \neq j}^{N} w_i \, w_j \, Cov(R_i, R_j)$$

subject to

$$\mu = \sum_{i=1}^{N} w_i E[R_i]$$

$$\sum_{i=1}^{N} w_i = 1$$



Solution method

We use Lagrange multipliers:

We express the system as

$$\min_{w} w^{'}\Sigma w$$
$$\text{subject to}$$
$$w^{'}M = m$$
$$w^{'}\mathbf{1} = 1$$

where Σ is the covariance matrix and $M = (\mathbb{E}(R_1), \cdots, \mathbb{E}(R_N))$.

We define the Lagrangian

$$L = w^{'}\Sigma w + \lambda_1(w^{'}M - m) + \lambda_2(w^{'}\mathbf{1} - 1)$$

and obtain the first order conditions

$$\Sigma w = -\frac{1}{2}(\lambda_1 M + \lambda_2 \mathbf{1})$$

The solution of the parametric first order conditions is

$$w = -\frac{1}{2}(\lambda_1 \Sigma^{-1} M + \lambda_2 \Sigma^{-1} \mathbf{1})$$

We substitute the above into the constraints and get the system of equations

$$m = w^{'} M = -\frac{1}{2}(\lambda_1 M^{'} \Sigma^{-1} M + \lambda_2 \mathbf{1}^{'} \Sigma^{-1} M) \tag{1}$$

$$1 = w^{'} \mathbf{1} = -\frac{1}{2}(\lambda_1 M^{'} \Sigma^{-1} \mathbf{1} + \lambda_2 \mathbf{1}^{'} \Sigma^{-1} \mathbf{1}) \tag{2}$$

We solve (1), (2) for the Lagrange multipliers

$$\begin{aligned} \lambda_1^* &= 2\frac{-m(\mathbf{1}^{'}\Sigma^{-1}\mathbf{1}) + \mathbf{1}^{'}\Sigma^{-1}M}{(M^{'}\Sigma^{-1}M)(\mathbf{1}^{'}\Sigma^{-1}\mathbf{1}) - (M^{'}\Sigma^{-1}\mathbf{1})(\mathbf{1}^{'}\Sigma^{-1}M)} \\ \lambda_2^* &= 2\frac{-M^{'}\Sigma^{-1}M + m(M^{'}\Sigma^{-1}\mathbf{1})}{(M^{'}\Sigma^{-1}M)(\mathbf{1}^{'}\Sigma^{-1}\mathbf{1}) - (M^{'}\Sigma^{-1}\mathbf{1})(\mathbf{1}^{'}\Sigma^{-1}M)} \end{aligned}$$

to obtain the optimal portfolio as

$$w^* = -\frac{1}{2}(\lambda_1^* \Sigma^{-1} M + \lambda_2^* \Sigma^{-1} \mathbf{1})$$

Introducing a riskless asset: The Tobin model

Assume for simplicity that we have two assets,

- A riskless asset of return r
- A risky asset of return R, such that $\mathbb{E}[R] = \mu_u$ and $Var(R) = \sigma^2$

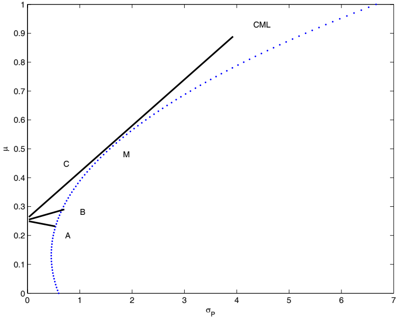
Then for a portfolio consisting of w percentage in the riskless asset we have that

$$\mu_P = wr + (1-w)\mu_u,$$
$$\sigma_P^2 = (1-w)^2\sigma^2$$

This forms a straight line in the $\sigma_P - \mu_P$ plane

$$\mu_P = r + (\mu_u - r)\frac{\sigma_P}{\sigma}$$

We now assume that the risky asset is not just a single asset but a Markowitz portfolio:



- The straight lines connecting the point $(0, r)$ and any point on the inverted parabola (efficient frontier) can be considered as portfolios consisting of combinations of the riskless asset and a fraction of a Markowitz portfolio.
- Of these lines the most interesting is one that starts from $(0, r)$ and is tangent to a point of the efficient frontier.
- This is the capital market line: It consists of portfolios that dominate on all Markowitz portfolio.
- The point M on the efficient frontier on which this line touches is called the market portfolio.

Portfolio separation theorem

Consider an investor that must decide on a portfolio consisting of a riskless asset and N risky assets in a market.

The optimal portfolio can be constructed in two stages:

- Find the Markowitz portfolios and among them single out the market portfolio M:
  - This is the one that maximizes

$$S := \frac{\mathbb{E}[R_p] - r}{\sigma_p} \quad \text{Sharpe ratio}$$

- Depending on your risk preferences allocate your wealth between M and the riskless asset constructing the portfolio

$$R = wr + (1 - w)R_M$$

Hence, you invest part of your wealth in a mutual fund R_M.

How can you construct the market portfolio?

Solve the optimization problem of maximizing the Sharpe ratio:

$$\max_{\{w_1,\cdots,w_N\}} = \frac{\mathbb{E}[r_P] - r}{\sigma_P}$$

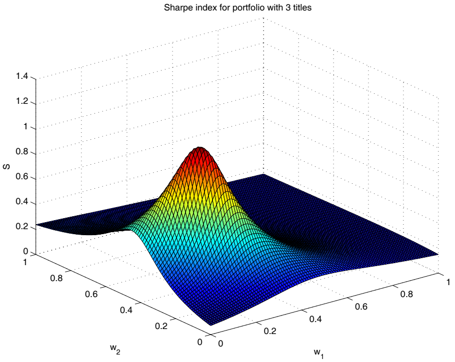
subject to

$$\sum_{i=1}^{N} w_i = 1$$

where

$$\mathbb{E}[r_P] = \sum_{i=1}^{N} w_i \mathbb{E}[R_i]$$

$$\sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}$$


Sharpe index for portfolio with 3 titles
S
1.4
1.2
1
0.8
0.6
0.4
0.2
0
w_2
1
0.8
0.6
0.4
0.2
0
w_1
0
0.2
0.4
0.6
0.8
1

The CAPM model

Under the assumption that all investor in the market are rational, each one selecting to divide her/his wealth between the riskless asset and a Markowitz portfolio, it can be shown that the all the returns of the assets in the market must be correlated to the market portfolio.

This is the celebrated CAPM model

$$R_i = \alpha_i + \beta_i R_M + \epsilon$$

Then,

$$Var(R_i) = \beta_i^2 (b_i R_M) + Var(\epsilon)$$

The first term is related to the risk of the market portfolio R_M – systematic risk (non differentiable risk)

β_i can be considered as a measure of systematic risk.

In practice we often consider as a proxy for R_M a large market index.

Alternative version for CAPM:

$$\mathbb{E}[R_i] - r = \frac{cov(R_i, R_M)}{\sigma_M^2}(R_M - r)$$

Why?

Consider a portfolio consisting of $a\%$ of the risky asset R_i and $(1-a)\%$ of the market portfolio R_M.

For this portfolio:

$$\begin{aligned} \mathbb{E}[R(a)] &:= \mathbb{E}[R_{P(a)}] = a\mathbb{E}[R_i] + (1-a)\mathbb{E}[R_M] \\ \sigma(a)^2 &:= \sigma^2_{P(a)} = a^2\sigma_i^2 + (1-a)^2\sigma_M^2 + 2a(1-a)Cov(R_i, R_M) \end{aligned}$$

- For $a = 0$ the portfolio $R(a)$ coincides with R_M.
- Varying a we obtain in the $\sigma - \mu$ plane a curve $L(a)$ which will be below the efficient frontier and tangent to it for $a = 0$.
- The derivative of the curve at $a = 0$ will concide with the slope of the capital market line i.e. $s_M = \frac{\mathbb{E}[R_M] - r}{\sigma_M}$

By elementary analytic geometry the slope of $L(a)$ at $a = 0$ is equal to

$$\left.\frac{\partial \mathbb{E}[R(a)]}{\partial \sigma(a)}\right|_{a=0} = \left.\frac{\frac{\partial \mathbb{E}[R(a)]}{\partial a}}{\frac{\partial \sigma(a)}{\partial a}}\right|_{a=0}$$

After some algebra

$$\begin{aligned}\frac{\partial \mathbb{E}[R(a)]}{\partial a} &= \mathbb{E}[R_i] - \mathbb{E}[R_M] \\ \frac{\partial \sigma(a)}{\partial a} &= \frac{1}{2}\frac{1}{\sqrt{\sigma(a)}}(2a\sigma_i^2 - 2(1-a)\sigma_M^2 + 2(1-2a)\sigma_{iM}\end{aligned}$$

so that

$$\left.\frac{\partial \mathbb{E}[R(a)]}{\partial \sigma(a)}\right|_{a=0} = \frac{\mathbb{E}[R_i] - \mathbb{E}[R_M]}{\frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}} = \frac{\mathbb{E}[R_M] - r}{\sigma_M}$$

where $\sigma_{i,M} = Cov(R_i, R_M)$ and in the last equality we substituted the slope of the capital market line.

The last relation yields that the expected return of asset i must satisfy

$$\mathbb{E}[R_i] = r + \beta_i\{\mathbb{E}[r_M] - r\}$$

where β_i, characterizing asset i, is given by

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = \frac{cov(R_i, R_M)}{Var(R_M}$$

This is the CAPM model.

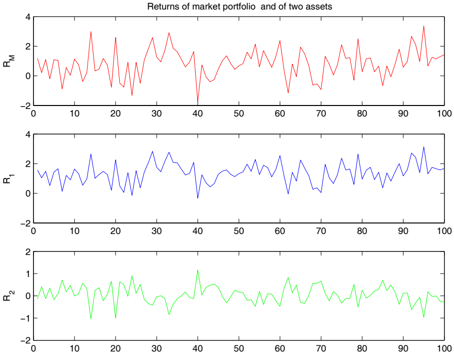
Estimators for the CAPM model

Given a proxy for the market portfolio we can estimate the CAPM model as in standard linear regression:

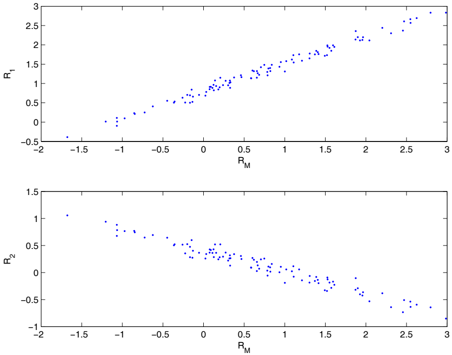
$$
\begin{aligned}
(\beta_i)_{est} &= \frac{\sum_{i=1}^{T}(R_{M,t} - (\mathbb{E}[R_M])_{est})R_{i,t}}{\sum_{t=1}^{T}(R_{M,t} - (\mathbb{E}[R_M])_{est})^2} \\
(a_i)_{est} &= (\mathbb{E}[R_i])_{est} - (\beta_i)_{est}(\mathbb{E}[R_M])_{est} \\
(\sigma_{ei}^2)_{est} &= \frac{1}{T-1}\sum_{i=1}^{T}[R_{i,t} - ((a_i)_{est} + (\beta_i)_{est}R_{M,t})]^2
\end{aligned}
$$

Example

Consider the following observations for a market index (market portfolio) and two assets



$$a_1 = 0,7988, \ \beta_1 = 0.6995$$
$$a_2 = 0.3988, \ \beta_2 = -0.4005$$



Portfolio performance indices

1. Sharpe ratio $S = \frac{\mathbb{E}[R_P]-r}{\sigma_P}$
2. Treynor index $T = \frac{\mathbb{E}[R_P]-r}{\beta_P}$
3. Jensen index a_P
4. Appraisal ratio $\frac{\alpha_P}{\sigma(\epsilon_P)}$ where $\sigma(\epsilon_P)$ is the non systematic risk of the portfolio

Multifactor market models

According to this model (Ross) the returns of all assets in the market are linearly dependent on K stochastic risk factors:

$$R_i = \mathbb{E}[R_i] + \beta_{i1} F_1 + \cdots + \beta_{iK} F_K + \epsilon_i, \quad i = 1, \cdots, N$$

or in compact form

$$R = \mathbb{E}[R] + \beta F + \epsilon$$

where $R \in \mathbb{R}^{1\times N}$, $\beta \in \mathbb{R}^{N\times K}$, $F \in \mathbb{R}^{K\times 1}$, $\epsilon \in \mathbb{R}^{N\times 1}$.

- βF systematic risk
- ϵ non systematic risk

Generalization to the CAPM (one factor) model.