Financial Mathematics

Option pricing in one period

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Definition

An asset is called a derivative if its value (equiv. payoff) depends on the value of another asset

Example (Call option)

A call option is a contract that allows the holder to buy an asset (e.g. a stock) at time T for a prescribed price K, no matter what the value of this asset is in the market.

If $S(T) > K$ then you exercise the option and payoff is $S(T) - K$
If $S(T) < K$ then you let the option expire unexercised and payoff is 0.
Payoff: $\max(S(T) - K, 0)$
You can buy it or sell it at any time $t < T$! What should be a fair price? Why use it?

Example (Put option)

A put option is a contract that allows the holder to sell an asset (e.g. a stock) at time T for a prescribed price K, no matter what the value of this asset is in the market.

If $S(T) < K$ then you exercise the option and payoff is $K - S(T)$
If $S(T) > K$ then you let the option expire unexercised and payoff is 0.
Payoff: $\max(K - S(T), 0)$
You can buy it or sell it at any time $t < T$! What should be a fair price?
Why use it?

There exists a wide variety of option on various underlying assets:

- Futures
- Swaps, Credit derivatives
- Asian options, Path dependent options
- Huge market:
  - Futures markets in June 2004 had outstanding positions of 53 tril. USD while in March 2008 of 81 tril USD
  - OTC markets (e.g. swaps, credit derivatives) in June 2004 has outstanding position of 220 tril. USD, end of 2007 of 596 tril USD, and 2009 of 615 tril USD!

A general definition of an option (derivative product)

Definition

A European option on an underlying asset with price $\{S(t) : \ t \in \mathbb{N}\}$ with expiry at T is an asset with payoff $F(S(T))$ at time T, where F is a given deterministic function.

A European option can be bought or sold in organized or OTC markets at any time $t < T$ for a price $P(t)$

A key problem in financial mathematics is to figure out what this price should be!

Spoiler: The fair price for an option is

$$P(t) = \mathbb{E}_Q[(1+r)^{-(T-t)} F(S(T)) \mid \mathcal{F}_t]$$

where

- $\mathcal{F}_t = \sigma(S(0), S(1), \cdots, S(t))$ the history of the market up to time t
- Q is the equivalent martingale measure

The formula

$$P(t) = \mathbb{E}_Q[(1+r)^{-(T-t)} F(S(T)) \mid \mathcal{F}_t]$$

makes sense:

- You discount the future payoff $F(S(T))$ payable at T to get its value at time t
- $F(S(T))$ is a random variable measurable with respect to $\mathcal{F}_T$ so you need its best prediction at time t (subject to the information structure $\mathcal{F}_t \subset \mathcal{F}_T$: $\mathbb{E}[\cdot \mid \mathcal{F}_t]$.

Two important questions arise:

- Why should we use the equivalent martingale measure Q (rather than the statistical measure P)?
  - Related to market equilibrium and absence of arbitrage – True for any market model
- How can we calculate the prediction $\mathbb{E}_Q[(1+r)^{-(T-t)} F(S(T)) \mid \mathcal{F}_t]$?
  - We need a statistical/probabilistic model for $S(T)$; Result is model dependent
  - Markov property for the underlying $\{S(t) \ : \ t \geq 0\}$ guarantees the existence of a function V such that $\mathbb{E}_Q[(1+r)^{-(T-t)} F(S(T)) \mid \mathcal{F}_t] = V(t, T, S(t))$ – Pricing function

Option pricing in the binomial model – One time period

We will address the above questions first for the one time period binomial model

$$S(1) = H_1 S(0), \quad P(H_1 = u) = 1 - P(H_1 = d) = p$$

Recall that for this model there exists an equivalent martingale measure Q, under which

$$Q(H_1 = u) = 1 - Q(H_1 = d) = q = \frac{1+r-d}{u-d}$$
$$0 < q < 1, \quad d < 1 + r < u$$

Under this measure

$$\mathbb{E}_Q[S(1)^*] = \mathbb{E}_Q[(1+r)^{-1} S(1)] = S(0)$$

Option pricing using replication

Use the market (riskless asset and underlying) to replicate the option.

- Time 0:
  - Riskless asset: Value 1
  - Underlying: Value $S(0) = S$
  - Option: Value P (to be determined)
- Time 1
  - Riskless asset: Value $(1 + r)$
  - Underlying: Value $S(1) = H_1 S(0)$ i.e. uS with probability p or dS with probability $1 - p$
  - Option: Value $F_1 = F(uS)$ with probability p or $F_2 = F(dS)$ with probability $1 - p$.

Aim: Construct a portfolio θ_0 in riskless asset and θ_1 in the underlying which at time 1, either in the up or down states has exactly the same performance as the option.

By absense of arbitrage, this portfolio at time 0 will have the same price as the option, hence revealing the unknown P.

- Time 0: $V(0) = \theta_0 + \theta_1 S$.
- Time 1
  - Up state: $V_1 = \theta_0(1+r) + \theta_1 u\, S = F_1$
  - Down state: $V_2 = \theta_0(1+r) + \theta_1 d\, S = F_2$

The above portfolio replicates the option.

Solving this system we obtain

$$\theta_1 = \frac{F_1 - F_2}{S(u-d)}, \quad \theta_0 = \dots \text{ (Exercise)}$$

The price of the option is the value of the replicating portfolio at $t = 0$, i.e.,

$$\begin{aligned} P &= \theta_0 + \theta_1\, S \\ &= \frac{1+r-d}{(1+r)(u-d)}\, F_1 + \frac{u-(1+r)}{(1+r)(u-d)}\, F_2 \end{aligned}$$

Defining

$$q = \pi_1 = \frac{1+r-d}{u-d}, \;\; \pi_2 = \frac{u-(1+r)}{u-d},$$

this can be expressed as

$$P = \pi_1 \frac{F_1}{1+r} + \pi_2 \frac{F_2}{1+r} = \mathbb{E}_Q[(1+r)^{-1}F]$$

Pricing by absense of arbitrage

Assume that at time $t = 0$ the price of the option is p.

The buyer of the option can select a portfolio at time $t = 0$ which consists of the option and the position $\bar{\theta}_0, \bar{\theta}_1$ in the riskless asset and the underlying respectively.

- At time $t = 0$: The value of this portfolio is $-p + \bar{\theta}_0 + \bar{\theta}_1 S$
- At time $t = 1$: The value of this portfolio is
    - In the up state: $F_1 + (1 + r)\bar{\theta}_0 + \bar{\theta}_1 u\, S$
    - In the down state: $F_2 + (1 + r)\bar{\theta}_0 + \bar{\theta}_1 d\, S$.

Can we use the option to speculate on the market, i.e. use the portfolio $(1, \bar{\theta}_0, \bar{\theta}_1)$ so that we create an arbitrage?

In other words, can we select $(1, \bar{\theta}_0, \bar{\theta}_1)$ so that its value is 0 at time $t = 0$ and positive in any state of the world at $t = 1$?

This leads to choosing $(\bar{\theta}_0, \bar{\theta}_1)$ as the solution of the system of inequalities

$$\begin{aligned} -p + \bar{\theta}_0 + \bar{\theta}_1 S &= 0 \\ F_1 + (1+r)\bar{\theta}_0 + \bar{\theta}_1 u\, S &\geq 0 \\ F_2 + (1+r)\bar{\theta}_0 + \bar{\theta}_1 d\, S &> 0? \end{aligned}$$

Suppose the existence of a such a pair $(\bar{\theta}_0, \bar{\theta}_1)$ and divide the second and third inequality $(1+r)$ (discounting),

$$\begin{aligned} -p + \bar{\theta}_0 + \bar{\theta}_1 S &= 0, \\ F_1^* + \bar{\theta}_0 + \bar{\theta}_1 u^* S &\geq 0, \\ F_2^* + \bar{\theta}_0 + \bar{\theta}_1 d^* S &> 0 \end{aligned}$$

Then multiply the second by $\pi_1 = \frac{1+r-d}{u-d}$ and the third by $\pi_2 = \frac{u-(1+r)}{u-d}$ and add to obtain

$$\mathbb{E}_Q[F^*] + \bar{\theta}_0 + \bar{\theta}_1 S > 0$$

This yields $p < \mathbb{E}_Q[F^*]$, which is a conditon for arbitrage.

Similarly, if $p > \mathbb{E}_Q[F^*]$ we also get arbitrage opportunities.

Hence, the only case where we do not have arbitrage opportunities is when $p = \mathbb{E}_Q[F^*]$.

Option pricing: Superhedging and pricing

An option is a contract exchanged between two agents: The buyer and the seller.

Seller:

- Time $t = 0$:
  - Receive the sum z to sell the contract
  - Create a portfolio (θ_0, θ_1) in the market that will offer returns so as to cover his/her obligations to the buyer of the contract:

$$V_0 := z = \theta_0 + \theta_1 S$$

- Time $t = 1$
  - Up state: Obligation to buyer $-F_1$:

$$V_1 := \theta_0(1 + r) + \theta_1 u\, S - F_1$$

  - Down state: Obligation to buyer $-F_2$:

$$V_1 := \theta_0(1 + r) + \theta_1 d\, S - F_2$$

Seller's superhedging portfolio

$$\begin{aligned} \theta_0 + \theta_1 S &= z, \\ \theta_0(1+r) + \theta_1 u\, S - F_1 &\geq 0, \\ \theta_0(1+r) + \theta_1 d\, S - F_2 &\geq 0 \end{aligned} \tag{1}$$

For large z this inequality is certainly true: But this is too big a price for the buyer to agree

Seller's price is the infimum of the set of prices that allow super hedging for the seller:

$$P_S = inf\{z \,:\, \exists\, \theta \,\text{such that (1) holds}\}$$

Take (1) divide the 2nd and 3rd by $(1+r)$ (discount) and subtract the result of each one from the first:

$$\theta_1(u^* - 1)S - F_1^* \geq -z \tag{2}$$
$$\theta_1(d^* - 1)S - F_2^* \geq -z \tag{3}$$

Multiply the first by $q = \pi_1 = \frac{1+r-d}{u-d}$, the second by $\pi_2 = 1 - q = \frac{u-(1+r)}{u-d}$ and add noting that $u^*\pi_1 + d^*\pi_2 = 1$, to obtain

$$z \geq \pi_1 F_1^* + \pi_2 F_2^* = \mathbb{E}_Q[F^*] \tag{4}$$

Note that in this inequality there is no sign of (θ_0, θ_1) but only z!

Any z which allows a superhedging strategy satisfies (4), hence also the infimum of the set of superhedging strategies, so

$$P_S \geq \mathbb{E}_Q[F^*].$$

An option is a contract exchanged between two agents: The buyer and the seller.

Buyer:

- Time $t = 0$:
  - Pay the sum Z to buy the contract
  - Create a lending portfolio (θ'_0, θ'_1) in the market that will allow him/her to raise the necessary funds Z to buy the contract:

$$V'_0 := -Z = \theta'_0 + \theta'_1 S$$

- Time $t = 1$
  - Up state: Payoff by option F_1:

$$V'_1 := \theta'_0(1 + r) + \theta'_1 u\, S + F_1$$

  - Down state: Payoff by option F_2:

$$V'_1 := \theta'_0(1 + r) + \theta'_1 d\, S + F_2$$

Buyer's superhedging portfolio

$$\begin{aligned} \theta_0' + \theta_1' S &= -Z, \\ \theta_0'(1+r) + \theta_1' u\, S + F_1 &\geq 0, \\ \theta_0'(1+r) + \theta_1' d\, S + F_2 &\geq 0 \end{aligned} \tag{5}$$

For small Z this inequality is certainly true: But this is too low a price for the seller to agree

Buyer's price is the supremum of the set of prices that allow super hedging for the buyer:

$$P_B = sup\{Z \,:\, \exists\, \theta' \,\text{such that (5) holds}\}$$

Working as above (discount 2nd and 3rd and subtract each one from the first) we have

$$\theta_1'(u^* - 1)S + F_1^* \geq Z$$
$$\theta_1'(d^* - 1)S + F_2^* \geq Z$$

and then multiplying 1st with $q = \pi_1$ and 2nd by $\pi_2 = 1 - q$ and adding we get

$$\pi_1 F_1^* + \pi_2 F_2^* = \mathbb{E}_Q[F^*] \geq Z \qquad (6)$$

Any price Z allowing the buyer to create a superhedging portfolio must satisfy (6), so the supremum of the set of such prices must also satisfy this inequality, hence:

$$P_B \leq \mathbb{E}_Q[F^*].$$

Proposition

The seller's price P_S and the buyer's price for an opton satisfy the inequality

$$P_B \leq \mathbb{E}_Q[F^*] \leq P_S.$$

This result (eventhough here was proved for the binomial model) is true for ANY market model

$[P_B, P_S]$ bid-ask spread.

Will these two prices ever coincide?

For the binomial model $P_B = P_S$

SELLER

- Set $\theta = (\theta_0, \theta_1)$ where $\theta_1 = \frac{F_1 - F_2}{S(u-d)}$ and $\theta_0 = ...$ (the hedging portfolio) and $z = \mathbb{E}_Q[F^*]$
- This choice satisfies inequality (1) hence $\mathbb{E}_Q[F^*]$ is an element of the set $\mathcal{A}_S := \{z \,:\, \exists\,\theta\,\text{such that}\,(1)\,\text{ holds}\}$.
- For any element z of this set we have $z \geq \mathbb{E}_Q[F^*]$ and the seller's price P_S is the inf of this set, hence the inf is attained therefore

$$P_S = \mathbb{E}_Q[F^*]$$

BUYER

- Set $\theta' = (\theta'_0, \theta'_1)$ where $\theta'_1 = -\frac{F_1 - F_2}{S(u-d)}$ and $\theta'_0 = -...$ (the hedging portfolio) and $Z = \mathbb{E}_Q[F^*]$
- This choice satisfies inequality (5) hence $\mathbb{E}_Q[F^*]$ is an element of the set $\mathcal{A}_B := \{Z \,:\, \exists\,\theta\,\text{such that}\,(5)\,\text{ holds}\}$.
- For any element Z of this set we have $Z \leq \mathbb{E}_Q[F^*]$ and the buyer's price P_B is the sup of this set, hence the sup is attained therefore

$$P_B = \mathbb{E}_Q[F^*]$$

Hence,

$$P_B = P_S = \mathbb{E}_Q[F^*].$$

In general $P_B \leq P_S$.

If the option (contingent claim) can be replicated in the market, then $P_B = P_S$.

In markets where any contingent claim can be replicated (complete markets) $P_B = P_S$!

This is equivalent to the existence of a unique equivalent martingale measure!