Επαναληπτικό Μάθημα Θεωρία Κατανομών

1) Αν $f(x_1, x_2) = 21x_1^2 x_2^3$, $0<x_1<x_2<1$ βρείτε την καταν. $f_{X_1|X_2=x_2}$

$0<x_2<1$

$$f(x_2) = \int_0^{x_2} 21x_1^2 x_2^3 \, dx_1 = 21x_2^3 \left. \frac{x_1^3}{3} \right|_0^{x_2} = 21x_2^3\left(\frac{x_2^3}{3} - 0\right) =$$

$$= 7x_2^6 \quad , \quad 0<x_2<1$$

$$f(x_1|x_2) = \frac{f(x_1,x_2)}{f(x_2)} = \frac{21x_1^2 x_2^3}{7x_2^6} = \frac{3x_1^2}{x_2^3} \quad , \quad 0<x_1<x_2 \quad , \quad 0<x_2<1$$

2) $(X_1, X_2, X_3) \sim$ Dirichlet

$$f(x_1,x_2,x_3) = c\, x_1^{\vartheta_1-1} x_2^{\vartheta_2-1} x_3^{\vartheta_3-1} (1-x_1-x_2-x_3)^{\vartheta_4-1} \qquad \begin{matrix} x_i>0 \\ x_1+x_2+x_3<1 \end{matrix}$$

$c>0$, $\vartheta_i>0$

Ν.Δ.Ο. $X_1 \sim$ Beta

<u>Λύση</u>

$$f(x_1,x_2) = \int_0^{1-x_1-x_2} f(x_1,x_2,x_3)\,dx_3 = \int_0^{1-x_1-x_2} c\,x_1^{\vartheta_1-1} x_2^{\vartheta_2-1} x_3^{\vartheta_3-1} (1-x_1-x_2-x_3)^{\vartheta_4-1} dx_3$$

$$\frac{x_3}{1-x_1-x_2} = y \Rightarrow dx_3 = (1-x_1-x_2)\,dy \qquad \begin{matrix} x_3=0 & y=0 \\ x_3=1-x_1-x_2 & y=1 \end{matrix}$$

$$\int_0^1 c\,x_1^{\vartheta_1-1} x_2^{\vartheta_2-1} [y(1-x_1-x_2)]^{\vartheta_3-1} [(1-x_1-x_2)(1-y)]^{\vartheta_4-1} (1-x_1-x_2)\,dy$$

$$c\,x_1^{\vartheta_1-1}\,x_2^{\vartheta_2-1}\,(1-x_1-x_2)^{\vartheta_3+\vartheta_4-1}\int_0^1 y^{\vartheta_3-1}(1-y)^{\vartheta_4-1}\,dy =$$

$$= c\,x_1^{\vartheta_1-1}(1-x_1-x_2)^{\vartheta_3+\vartheta_4-1}\,x_2^{\vartheta_2-1}\cdot B(\vartheta_3,\vartheta_4) \quad , \; 0<x_1+x_2<1, \; x_1,x_2>0$$

$$f(x_1)=\int f(x_1,x_2)\,dx_2 = \int_0^{1-x_1} c\,x_1^{\vartheta_1-1}(1-x_1-x_2)^{\vartheta_3+\vartheta_4-1}\,x_2^{\vartheta_2-1}\,B(\vartheta_3,\vartheta_4)\,dx_2 =$$

$$= \underbrace{B(\vartheta_3,\vartheta_4)\,c\,x_1^{\vartheta_1-1}}_{K}\int_0^{1-x_1}(1-x_1-x_2)^{\vartheta_3+\vartheta_4-1}\,x_2^{\vartheta_2-1}\,dx_2$$

$$\frac{x_2}{1-x_1}=z \Rightarrow dx_2=(1-x_1)\,dz \qquad x_2=0: \; z=0 \qquad x_2=1-x_1: \; z=1$$

$$K\int_0^1 (1-x_1-z(1-x_1))^{\vartheta_3+\vartheta_4-1}\left[(1-x_1)z\right]^{\vartheta_2-1}(1-x_1)\,dz =$$

$$=K\int_0^1 (1-x_1)^{\vartheta_2+\vartheta_3+\vartheta_4-1}(1-z)^{\vartheta_3+\vartheta_4-1}\,z^{\vartheta_2-1}\,dz =$$

$$=K(1-x_1)^{\vartheta_2+\vartheta_3+\vartheta_4-1}\int_0^1 (1-z)^{\vartheta_3+\vartheta_4-1}\,z^{\vartheta_2-1}\,dz = K\,B(\vartheta_2,\vartheta_3+\vartheta_4)\,(1-x_1)^{\vartheta_2+\vartheta_3+\vartheta_4-1}$$

$$= c\,B(\vartheta_3,\vartheta_4)\,B(\vartheta_2,\vartheta_3+\vartheta_4)\,x_1^{\vartheta_1-1}(1-x_1)^{\vartheta_2+\vartheta_3+\vartheta_4-1} \quad , \; x_1\in(0,1)$$

3) $f_{X,Y}(x,y) = \begin{cases} 3x, & 0 \le y \le x \le 1 \\ 0, & \text{αλλού} \end{cases}$

Ζητείται: $Z = X \cdot Y$, $U = X/Y$

Λύση

$\left.\begin{array}{l} z = xy \\ u = \frac{x}{y} \end{array}\right\} \Rightarrow \left.\begin{array}{l} z = uy^2 \\ x = uy \end{array}\right\} \Rightarrow \left.\begin{array}{l} y^2 = \frac{z}{u} \\ x = uy \end{array}\right\} \Rightarrow \left.\begin{array}{l} y = \sqrt{\frac{z}{u}} \\ x = u\sqrt{\frac{z}{u}} \end{array}\right\} \Rightarrow \begin{array}{l} x = \sqrt{zu} \\ y = \sqrt{\frac{z}{u}} \end{array}$

$z, u > 0$

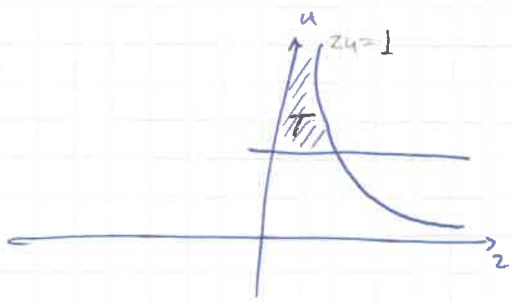
$$J = \begin{vmatrix} \frac{\sqrt{u}}{2\sqrt{z}} & \frac{1}{2\sqrt{zu}} \\ \frac{\sqrt{z}}{2\sqrt{u}} & -\sqrt{z}\,\frac{1}{2u\sqrt{u}} \end{vmatrix} = -\frac{\sqrt{\cancel{u}z}}{4u\sqrt{\cancel{u}z}} - \frac{\sqrt{z}}{4\sqrt{zu^2}} = \frac{-\sqrt{z}-\sqrt{z}}{4u\sqrt{z}} = \frac{-2\sqrt{z}}{4u\sqrt{z}} = -\frac{1}{2u} \neq 0$$

$T = \{(z,u) \in \mathbb{R}^2 : z = xy ,\ u = \frac{x}{y} ,\ (x,y) \in S\} = \{(z,u) \in \mathbb{R}^2 : x = \sqrt{zu} ,\ y = \sqrt{\frac{z}{u}}\}$

$= \{(z,u) \in \mathbb{R}^2 : 0 < \sqrt{zu} < \sqrt{\frac{z}{u}} < 1\} = \{(z,u) \in \mathbb{R}^2 : 0 < \sqrt{\frac{z}{u}} < \sqrt{zu} < 1\}$

$= \{(z,u) \in \mathbb{R}^2 : 0 < \frac{z}{u} < zu < 1\}$

$\frac{z}{u} < zu \Rightarrow u^2 > 1 \quad u > 1 \qquad zu < 1$



$f(z,u) = 3\sqrt{zu}$, $zu \in T$

i) $f(z) = \int_1^{\frac{1}{z}} 3\sqrt{zu}\, du$

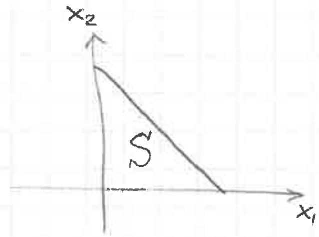
ii) $f(u) = \int_0^{\frac{1}{u}} 3\sqrt{zu}\, dz$

Άσκηση $f(x_1,x_2)=\begin{cases} c(x_1+x_2), & 0<x_1<1,\ 0<x_2<1,\ 0<x_1+x_2<1 \\ 0, & \text{αλλού} \end{cases}$

(i) Δείξτε ότι $c=3$

(ii) $f(x_2 | x_1)$, (iii) $Cov(X_1,X_2)$, (iv) $P(X_2>X_1 \,/\, X_1>\frac{1}{4})$

Λύση



$$\iint f(x_1,x_2)\,dx_1\,dx_2=1 \quad \ldots \quad c=3$$

ii) $$f(x_1)=\int_0^{1-x_1}3(x_1+x_2)\,dx_2=\int_0^{1-x_1}3(x_1+x_2)\,dx_2=3x_1(1-x_1)$$

$$+\frac{3}{2}x_2^2\Big|_0^{1-x_1}=3x_1-3x_1^2+\frac{3}{2}(1-x_1)^2=\frac{3}{2}(1-x_1^2), \quad x_1\in(0,1)$$

$$f(x_2/x_1)=\frac{3(x_1+x_2)}{\frac{3}{2}(1-x_1^2)}=\frac{2(x_1+x_2)}{1-x_1^2}, \quad 0<x_2<1-x_1$$
$(x_1\in(0,1))$

iii) $Cov(X_1,X_2)=E(X_1X_2)-EX_1EX_2$

$$EX_1X_2=\iint_S 3x_1x_2(x_1+x_2)\,dx_1dx_2=\int_0^1\left[\int_0^{1-x_1}3x_1x_2(x_1+x_2)\,dx_2\right]dx_1=$$

$$=\int_0^1\left[\int_0^{1-x_1}3x_1^2x_2\,dx_2+\int_0^{1-x_1}3x_1x_2^2\,dx_2\right]dx_1=\int_0^1\left[3x_1^2\frac{x_2^2}{2}\Big|_0^{1-x_1}+x_1x_2^3\Big|_0^{1-x_1}\right]dx_1=$$

$$=\int_0^1\left(\frac{3x_1^2}{2}(1-x_1)^2+x_1\cdot(1-x_1)^3\right)dx_1=\int_0^1\left[\frac{3}{2}x_1^2(1-x_1)^2+x_1(1-x_1)^3\right]dx_1=\ldots$$

$$EX_1=\int_0^1\frac{3}{2}(1-x_1^2)\,dx_1$$

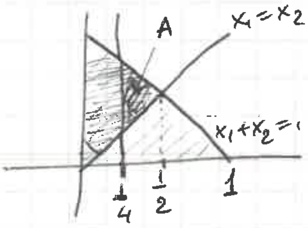
$$EX_2=\iint x_2\,3(x_1+x_2)\,dx_1dx_2 \quad ή$$

$\int x_2 f(x_2)\,dx_2$ πρώτα τη $f(x_2)$

iv) $P(X_2 > X_1 \mid X_1 > \frac{1}{4})$

$$P(X_1 > \tfrac{1}{4}) = \int_{1/4}^{1} \frac{3}{2}(1-x_1^2)\,dx_1 = \frac{3}{2}\left(1-\frac{1}{4}\right) - \frac{1}{2}x_1^3 \Big/_{\frac{1}{4}}^{1} = \frac{3}{2}\,\frac{3}{4} - \frac{1}{2}\left(1-\frac{1}{64}\right) =$$

$$= \frac{9}{8} - \frac{1}{2} + \frac{1}{128} = \frac{9\cdot 16 - 64 + 1}{128} = \frac{81}{128}$$



$$P(X_2 > X_1 \mid X_1 > \tfrac{1}{4}) =$$

$$= \frac{P(X_2 > X_1,\ X_1 > \frac{1}{4})}{P(X_1 > \frac{1}{4})}$$

$$= \frac{\iint_A f(x_1,x_2)\,dx_1\,dx_2)}{\frac{81}{128}}$$

$$\iint_A f(x_1,x_2)\,dx_1\,dx_2 = \int_{1/4}^{1/2}\left[\int_{x_1}^{1-x_1} 3(x_1+x_2)\,dx_2\right]dx_1 = \int_{1/4}^{1/2}\left[3x_1 x_2\Big/_{x_1}^{1-x_1} + \frac{3}{2}x_2^2\Big/_{x_1}^{1-x_1}\right]dx_1$$

$$= \int_{1/4}^{1/2}\left[3x_1(1-x_1-x_1) + \frac{3}{2}\left[(1-x_1)^2 - x_1^2\right]\right]dx_1 = \int_{1/4}^{1/2}\left[3x_1 - 6x_1^2 + \frac{3}{2}(1+\cancel{x_1^2} - 2x_1 - \cancel{x_1^2})\right]dx_1 =$$

$$= \int_{1/4}^{1/2}\left[3x_1 - 6x_1^2 + \frac{3}{2}(1-2x_1)\right]dx_1 = \int_{1/4}^{1/2}\left[\cancel{3x_1} - 6x_1^2 + \frac{3}{2} - \cancel{3x_1}\right]dx_1 = -2x_1^3\Big/_{\frac{1}{4}}^{\frac{1}{2}} + \frac{3}{2}\left(\frac{1}{2}-\frac{1}{4}\right) =$$

$$= -2\left(\frac{1}{8} - \frac{1}{64}\right) + \frac{3}{2}\,\frac{1}{4} = \frac{3}{8} - 2\,\frac{7}{64} = \frac{3}{8} - \frac{14}{64} = \frac{10}{64} = \frac{5}{32}$$

Άσκηση

$F_X(x) = 1 - e^{-x^2/2}$
$x > 0$

(α) f

(β) EX

(γ) Διάμεσο της X

(δ) Δείγμα $n=5$ βρείτε τη δειγμ. διάμεσο και την μέση της τιμή

Λύση

α) $f(x) = F'_X(x) = +e^{-x^2/2} \cdot x \Rightarrow f(x) = xe^{-x^2/2}, \quad x > 0$

β) $EX = \int_0^\infty x^2 e^{-x^2/2} dx = \ldots$ με $y = x^2$ και Γάμμα ολοκλ. $= \frac{\sqrt{2\pi}}{2}$

ή

$EX = \sqrt{2\pi} \frac{1}{\sqrt{2\pi}} \int_0^\infty x^2 e^{-x^2/2} dx = \sqrt{2\pi} \cdot \frac{1}{2} Var(Z)$ όπου $Z \sim N(0,1)$

$= \frac{\sqrt{2\pi}}{2}$

(γ) $F(m) = \frac{1}{2} \Rightarrow 1 - e^{-m^2/2} = \frac{1}{2} \Rightarrow e^{-m^2/2} = \frac{1}{2} \Rightarrow -\frac{m^2}{2} = -\log 2 \Rightarrow m^2 = 2\log 2 \Rightarrow m = \sqrt{2\log 2}$

(δ) $n=5 \quad r=3 \quad f_{Y_3}(y) = \frac{5!}{2!\,2!} \left(F_X(y)\right)^2 \left(1 - F_X(y)\right)^2 f_X(y)$

$= \frac{5!}{2!2!} \left(1 - e^{-y^2/2}\right) e^{-y^2/2} \cdot y e^{-y^2/2}, \quad y > 0$

$EY = \int y f(y) dy$

$e^{-y^2/2} = t \Rightarrow -\frac{y^2}{2} = \log t \Rightarrow -y^2 = 2\log t$

$y^2 = -2\log t \Rightarrow \cancel{2} y\, dy = -\cancel{2}\frac{1}{t} dt$

$y = 0 \quad t = 1$

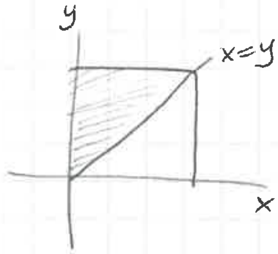
$y \to \infty \quad t = 0$

$\int_1^0 \frac{5!}{2!2!} (1-t) t^2 \left(-\frac{1}{t}\right) dt = \frac{1 \cdot 2 \cdot 3 \cdot \cancel{4} \cdot 5}{\cancel{4}} \int_0^1 (1-t) t \, dt$

$= 30 \, B(2,2) = 30 \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} = \frac{6 \cdot 5 \cdot 1 \cdot 1}{3 \cdot 2} = 5$

Άσκηση

$f(x,y) = cx^2 y \qquad 0 \le x \le y \le 1$



i) $c = ;$

ii) Είναι οι X, Y ανεξάρτ. ;

iii) $f(x|y)$

iv) Για ποια τιμή της Y ~~f(x|y)~~ $E(X|Y=y) = EX;$

Λύση

i) $c = 15$

ii) $f(x) = \int_x^1 c x^2 y \, dy = 15 \frac{x^2}{2}(1-x^2), \quad x \in (0,1)$

$f(y) = \int_0^y c x^2 y \, dx = 15\left(\frac{x^3}{3} \Big/_0^y\right) y = \frac{15y}{3}(y^3) = 5y^4, \quad y \in (0,1)$

ΔΕΝ είναι ανεξ.

iii) $f(x|y) = \frac{f(x,y)}{f(y)} = \frac{15x^2 y}{5y^4} = 3\frac{x^2}{y^2}, \quad 0 < x < y, \quad y \in (0,1)$

iv) $E(X|Y=y) = \int x f(x|y) dx = \int_0^y x \cdot 3 \frac{x^2}{y^2} dx = \frac{3}{y^2} \frac{x^4}{4} \Big/_0^y =$

$= \frac{3}{y^2} \frac{1}{4}(y^4 - 0) = \frac{3}{4} y^2$

$EX = \int_0^1 \frac{15}{2} x^3 (1-x^2) dx = \frac{15}{2} \int x^3 dx - \frac{15}{2} \int x^5 dx = \frac{15}{8} x^4 \Big/_0^1 - \frac{15}{12} x^6 \Big/_0^1 =$

$= \frac{15}{8} - \frac{15}{12} = \frac{45-30}{24} = \frac{15}{24}$

$\frac{3}{4} y^2 = \frac{15}{24} \Rightarrow y^2 = \frac{\cancel{15}^{5}}{\cancel{24}_{6}} \frac{4}{\cancel{3}} \Rightarrow y^2 = \frac{5}{6} \Rightarrow y = \sqrt{\frac{5}{6}} \quad \in (0,1)$