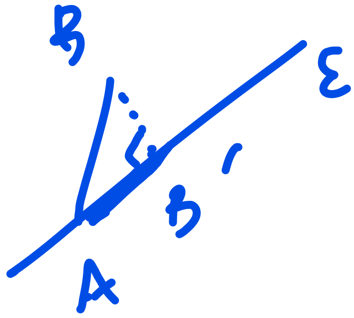
9 Μαρτίου

① ΠΙΝΑΚΑΣ ΠΡΟΒΟΛΗΣ

(Projection Matrix)



$P_A = A(A^TA)^{-1}A^T$.

$A^{m\times n}$

$R(A)$ = χώρος στηλών. $P_A \cdot \vec{u} = \hat{u}$

$\hat{u} \in R(A)$

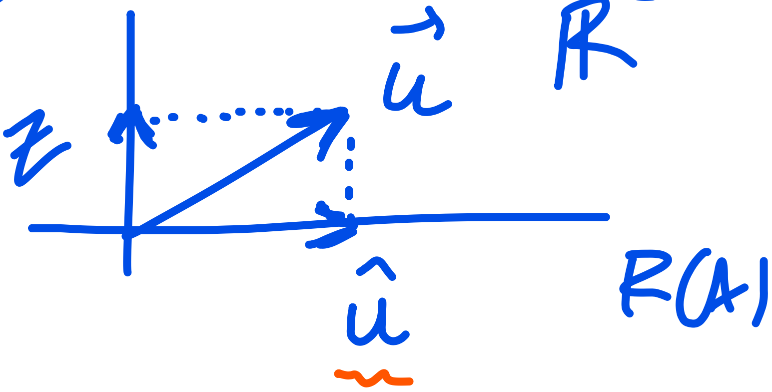
A γνωστό, P_A? (A^{-1} δεν υπάρχει)

Θ. Έστω $\vec{u} \in \mathbb{R}$. A, $R(A) \subset \mathbb{R}^n$

$\vec{u} = \hat{u} + z$ $(\hat{u} \perp z)$

$R(A)$ $\quad$ $R(A)^{\perp}$

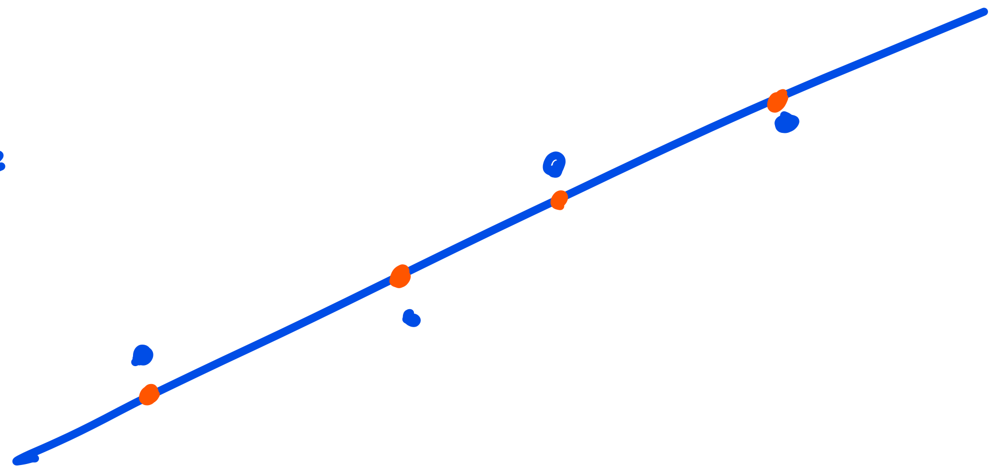
$P_A u = \hat{u}$



$\boxed{P \cdot \vec{u} = \hat{u}}$

$(z = u - \hat{u})$

Av

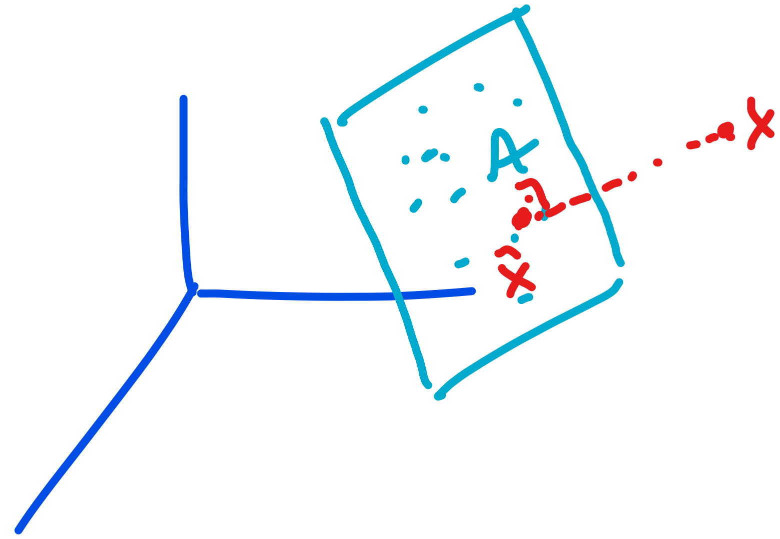


ext to $\mathbb{R}^4$

θ. $A \subseteq V$ $\quad (V = S.x.)$ $\quad P_A \to A$

$x \in V$: Av $P x = \hat{x}$: $\|\hat{x} - y\| \le \|x - y\|$

$\mathbb{R}^3$



$\forall y \in A$:

$P_A x = \hat{x}$

Ex

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \qquad \text{rank}(A) = 2$$

$(A : \mathbb{R}^2 \to \mathbb{R}^3)$

P_A ?

$$A\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$R(A)$

$$A\vec{u} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 2x+y \\ 3x+y \end{bmatrix} = \begin{bmatrix} k \\ x+k \\ 2x+k \end{bmatrix}$$

και $P_A = A(A^TA)^{-1} \cdot A^T$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -\frac{1}{6} & 1/3 & 5/6 \end{bmatrix} = P_A.$$

Τυχαίο διάνυσμα: $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$?

$$P_A \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \ldots \begin{pmatrix} -1/3 \\ 2/3 \\ 5/3 \end{pmatrix}$$

$R(A)$: $\begin{pmatrix} k \\ x+k \\ 2x+k \end{pmatrix}$ ΝΑΙ

$x = 1$

$k = -1/3$

Ιδιότητες P:

$$\underbrace{P^2 = P}, \quad P^T = P$$

$$P(Pu) = P\hat{u} = \hat{u}$$

$\underline{\underline{\Theta}}$ P πίνακας προβολής

$y \in \mathbb{R}^n$ τυχαίο.

$P^2 = P$

$P^T = P$

$\underline{Py = \hat{y}}, \quad y = \hat{y} + z$

$\boxed{\text{Ισχύει} \quad \hat{y} \perp z}$

~~Απόδ~~: $\langle z, \hat{y} \rangle = (y - \hat{y}) \cdot Py =$

$z \cdot \hat{y}$

$$y \cdot (Py) - \hat{y} \cdot Py =$$

$$y^T P y - (Py)^T P y =$$

$$y^T P y - y^T P^T P y =$$

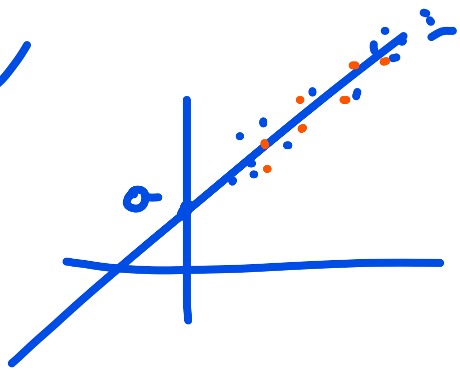
$$y^T P y - y^T P^2 y =$$

$$y^T P y - y^T P y = 0.$$

Άρα $\hat{y} \perp z$.

(B) Ευθεία ελαχίστων τετραγώνων

$$\boxed{y = a + bx}$$



Best fit line

$$\boxed{y = a + bx}$$

Δεδομένα (x,y) $\boxed{y = Ax}$ $\Rightarrow$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_4 \end{bmatrix} = \overset{A}{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_4 \end{bmatrix}} \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow$$

$$Ax = y \Rightarrow \boxed{\underline{A^T} A x = A^T y}$$

(Normal) $\rightarrow$ ΚΑΝΟΝΙΚΗ ΕΞΙΣΩΣΗ της $Ax = y$

$$\boxed{\left.\begin{aligned} &A : 4 \times 2 \\ &A^T = 2 \times 4 \end{aligned}\right\} A^T A = 2 \times 2}$$

$$\hat{x} = (A^T A)^{-1} A^T y$$

βέλτιστη λύση

Άρα

$$A\hat{x} = \underbrace{A(A^T A)^{-1} A^T}_{} y.$$

$$A\hat{x} = P_A y = y_1$$

$$Ax = y \quad \Longrightarrow \quad \boxed{A\hat{x} = y_1}$$

μη αντιστρέψιμος

$\in R(A)$

Πχ 1

x	1	2	3	4
y	23	27	30	34

$$\boxed{y = a + bx} : \begin{cases} a + b = 23 \\ a + 2b = 27 \\ a + 3b = 30 \\ a + 4b = 34 \end{cases}$$

Δεν έχει λύση!

$$A : \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 23 \\ 27 \\ 30 \\ 34 \end{bmatrix}$$

$\boxed{Ax = b}$ x?

$\hat{x}$?

$$A \cdot \vec{x} = \vec{y} \Rightarrow$$

$$\underbrace{\underset{2\times 4}{A^T} \; \underset{4\times 2}{A}}_{2\times 2} \vec{x} = \underset{2\times 4}{A^T} \, \underset{4\times 1}{y} \Rightarrow$$

$$\begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \vec{x} = \begin{bmatrix} 114 \\ 303 \end{bmatrix} \Rightarrow$$

$\det A^T A = 20$ ✓

$$\vec{x} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} 114 \\ 303 \end{bmatrix} = \begin{bmatrix} 19.5 \\ 3.6 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$y = 19.5 + 3.6x$$

$$\text{M.E.T.} = \begin{bmatrix} 19{,}5 \\ 3{,}6 \end{bmatrix}$$

$$\boxed{Ax=y}\ ? \qquad \hat{x}\ ?$$

Λ.Ε.Τ.

$$\Rightarrow A^TAx = A^Ty \Rightarrow \hat{x} = (A^TA)^{-1}\cdot A^Ty$$

<u>Τι λύνει</u>? ΟΧΙ ΤΟ $Ax=y$

Λύνει το: $A\hat{x} = A(A^TA)^{-1}A^Ty = P_A y = y_1$

$\boxed{Ax=y_1}$ Αυτό λύνει

Προσεγγιστικά.

$$y = y_1 + y_2$$

$$y_1 \downarrow R(A) \qquad y_2 \downarrow R(A)^\perp$$