23 Μαρτ.

ΙΔΙΟΤΙΜΕΣ : λ_i

ΙΔΙΟΔΙΑΝΥΣΜΑΤΑ : $\vec{u}$

(eigenvalues)

(eigenvectors)

$A^{n \times n}$

Ιδιοτιμές $\lambda_i \in \mathbb{R}$ ή $\mathbb{C}$

$A^{3\times 3}$: $\longrightarrow$ 3 ιδιοτιμές (λ_i)

$$\boxed{\underbrace{A\vec{u}}_{\text{ιδιοδιάνυσμα}} = \underbrace{\lambda}_{\text{ιδιοτιμή}}\vec{u}}$$

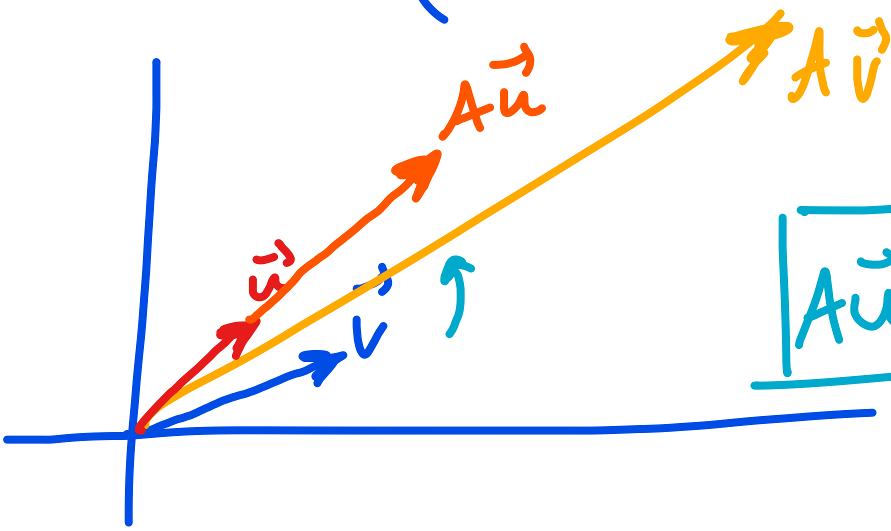
$$A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\Downarrow$$

$$\lambda = 3, \quad \lambda = -2$$

$$\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{à } \lambda_2 o$$

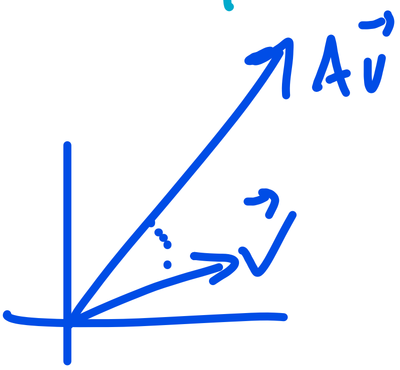
$$A \cdot \vec{v} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

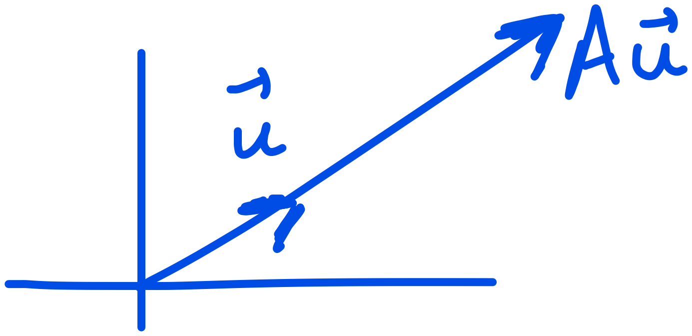


$$\boxed{A\vec{u} \,/\!/\, \vec{u}}$$

$$A\vec{u} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

Γεωμετρικά





$$\boxed{A\vec{u} = \lambda\vec{u}} \iff A\vec{u} - \lambda\vec{u} = \vec{0} \Rightarrow$$

$$(A - \lambda I)\cdot\vec{u} = \vec{0} \iff \boxed{\vec{u} \neq \vec{0}}$$

$$\boxed{\det(A - \lambda I) = 0}$$

Πχ1:

$$A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \quad (2\times 2)$$

- $A - \lambda I = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix}$

• $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow$

$(2-\lambda)(-1-\lambda) - 4 = 0 \Rightarrow$

$-2 - 2\lambda + \lambda + \lambda^2 - 4 = 0 \Rightarrow \lambda^2 - \lambda - 6 = 0$

$\lambda_1 = 3, \quad \lambda_2 = -2.$

(Ιδιοτιμές).

$\sigma(A) = 3, -2$

?

Ιδιοδιανύσματα (A) $\lambda = 3$

$\boxed{\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}}$

$A\vec{u} = \lambda \vec{u} \Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow$

$\begin{cases} 2x + 2y = 3x \\ 2x - y = 3y \end{cases} \rightarrow \begin{cases} -x + 2y = 0 \Rightarrow \\ \cancel{2x - 4y = 0} \end{cases}$

$x=2y \qquad \vec{u}=\begin{pmatrix}2y\\y\end{pmatrix}=\begin{pmatrix}2k\\k\end{pmatrix} \quad k\in\mathbb{R}$

$\lambda=3 \longrightarrow \vec{u}=\begin{pmatrix}2k\\k\end{pmatrix}, \; k\in\mathbb{R}$ πχ $\begin{pmatrix}2\\1\end{pmatrix}$

Ιδιόχωρος της $\lambda=3 \longrightarrow \vec{u}$

B) $\lambda=-2$: $\boxed{A\vec{v}=-2\vec{v}}$

$$\begin{bmatrix}2 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=-2\begin{bmatrix}x\\y\end{bmatrix} \Longrightarrow$$

$$\begin{cases}2x+2y=-2x\\2x-y=-2y\end{cases} \Rightarrow \begin{cases}\cancel{4x+2y=0}\\2x+y=0\end{cases}$$

$$\Longrightarrow y=-2x$$

$$\vec{V} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -2x \end{pmatrix} = \begin{pmatrix} t \\ -2t \end{pmatrix},\ t \in \mathbb{R}$$

ή $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\underline{\lambda = -2} \longrightarrow \vec{V} = \begin{pmatrix} t \\ -2t \end{pmatrix},\ t \in \mathbb{R}$$

$$\boxed{\begin{aligned} V_A(-2) &= \begin{pmatrix} t \\ -2t \end{pmatrix},\ t \in \mathbb{R} \\ V_A(3) &= \begin{pmatrix} 2k \\ k \end{pmatrix},\ k \in \mathbb{R} \end{aligned}}$$

ΙΔΙΟΧΩΡΟΙ ΤΟΥ A.

$n \times 2$

3x3 πίνακας :

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

• Ιδιοτιμές: $\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow$

$$\begin{array}{ccccc} + & + & + & & \\ 2-\lambda & 1 & 0 & 2-\lambda & 1 \\ 0 & 1-\lambda & -1 & 0 & 1-\lambda \\ 0 & 2 & 4-\lambda & 0 & 2 \\ - & - & - & & \end{array}$$

(Sarrus)

$$(2-\lambda)(1-\lambda)(4-\lambda) + 2(2-\lambda) = 0 \Rightarrow$$

$$(2-\lambda)\left[(1-\lambda)(4-\lambda) + 2\right] = 0 \Rightarrow$$

$$(2-\lambda)(\lambda^2 - 5\lambda + 6) = 0$$

$\lambda = 2$, $\lambda = 2$, $\lambda = 3$

$\lambda = 2$ διπλή, $\lambda = 3$

$\sigma(A) = \{2, 3\}$.

$\lambda = 2$: Αλγεβρική πολλαπλότητα $= 2$

Σχόλιο

• Ιδιοδιανύσματα: $\lambda = 3$: $\boxed{A\vec{u} = 3\vec{u}}$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow$$

$$\begin{cases} 2x + y = 3x \\ y - z = 3y \\ 2y + 4z = 3z \end{cases} \Rightarrow \begin{cases} y = x \\ z = -2y \\ \cancel{z = 2y} \end{cases}$$

$$\vec{u} = \begin{pmatrix} y \\ y \\ -2y \end{pmatrix} = \begin{pmatrix} k \\ k \\ -2k \end{pmatrix} \quad k \in \mathbb{R}$$

$$\text{πχ} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

(διάσταση 1 ο ιδιόχωρος)

- $\lambda = 2$: $A\vec{u} = 2\vec{u} \Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\Rightarrow \begin{cases} 2x + y = 2x \Rightarrow \boxed{y = 0} \\ y - z = 2y \leftarrow \boxed{z = 0} \\ 2y + 4z = 2z \leftarrow \quad 0 = 0 \checkmark \end{cases}$$

$$\vec{v} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix},\ t \in \mathbb{R}.$$

Διάσταση 1

ΑΡΑ $\quad A \xrightarrow{\lambda_i} 2, 3$ διπλή

Αντίστοιχα: $\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k \\ k \\ -2k \end{pmatrix}$

(Γ.Α.) $t, k \in \mathbb{R}$

~~ΠΑΡΑΤΗΡΗΣΗ~~:

$A\ (3 \times 3)$, 3 ιδιοτιμές

$\longrightarrow$ 2 ιδιοδιανύσμ.

- Αν $A = A^T \to \vec{u}, \vec{v} : \vec{u} \perp \vec{v}$

Θ. Αν $\lambda = 0$ = ιδιοτιμή $\Longleftrightarrow$

A^{-1} δεν υπάρχει.

$\bullet \quad \det(A - \lambda I) = 0$ $\quad \nexists$

$\lambda = 0$

$\det(A) = 0 \Rightarrow$

A^{-1}

δεν υπάρχει

Αν $\det(A) = 0$, $\rightarrow \lambda I = 0$ $\quad \lambda = 0$

Ιδιότητες

Ιδιοτιμών - Ιδιοδιανυσμάτων:

- $\sigma(A) = \sigma(A^T)$

- Αν λ ιδιοτιμή του $A \Rightarrow \lambda^k = $ ιδιοτ. A^k

(ίδιο ιδιοδιάνυσμα)

Γιατί:

Έστω $\boxed{A\vec{u} = \lambda \vec{u}}$ → ιδιοτιμή

πχ $A^3 u = A^2 A u = A^2 \lambda u =$

$\lambda A \cdot A u = \lambda \cdot A \lambda u =$

$\lambda^2 A \vec{u} = \lambda^3 \vec{u}$ ✓

(Αν A^{-1} υπάρχει)

• Αν λ ιδιοτιμή $A \Rightarrow \frac{1}{\lambda}$ ιδιοτιμή A^{-1}

$A\vec{u} = \lambda \vec{u} \Rightarrow (\cdot A^{-1})$

$I u = \lambda A^{-1} u \Rightarrow \frac{1}{\lambda} \vec{u} = A^{-1} u$

• Όμοιοι (A, B)

$B = P^{-1} A P$

για κάποιον P.

$\Rightarrow$ ίδιες ιδιοτιμές

- $A = A^T \Rightarrow \lambda_i \in \mathbb{R}$

  $A^T = -A^T \Rightarrow \lambda_i \in \mathbb{C}$

- Q ορθογώνιος $Q^T = Q^{-1}$

  $\Rightarrow |\lambda_i| = 1$

$n \times 3$:

$$A = \begin{bmatrix} 5 & 7 \\ 2 & -4 \end{bmatrix}$$

λ_i ? $\vec{u}$?

(A) Ιδιοτιμές $|A - \lambda I| = 0 \Rightarrow$

$$\begin{vmatrix} 5-\lambda & 7 \\ -2 & -4-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(-4-\lambda)+14=0$$

$$\Rightarrow -20-5\lambda+4\lambda+\lambda^2+14=0$$

$$\lambda^2-\lambda-6=0 \quad \begin{matrix} \nearrow -2 \\ \searrow 3 \end{matrix}$$

B Για $\lambda=-2$ $\quad A\vec{u}=-2\vec{u} \Rightarrow$

$$\begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -2\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow$$

$$\begin{cases} 5x+7y=-2x \\ -2x-4y=2y \end{cases} \Rightarrow \begin{cases} 7x+7y=0 \\ -2x=2y \Rightarrow y=-x \end{cases}$$

$$\vec{u}=\begin{pmatrix} k \\ -k \end{pmatrix}$$

Γ Για $\lambda=3$:

$$\begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix} \to$$

$$\begin{cases} 5x + 7y = 3x \\ -2x - 4y = 3y \end{cases} \Rightarrow \begin{cases} \cancel{2x + 7y = 0} \\ -2x = 7y \end{cases}$$

$$y = -\frac{2}{7} x$$

$$\vec{v} = \begin{pmatrix} x \\ -\frac{2}{7} x \end{pmatrix} = \begin{pmatrix} t \\ -\frac{2}{7} t \end{pmatrix} \quad \overset{n7}{\begin{pmatrix} 7 \\ -2 \end{pmatrix}}$$