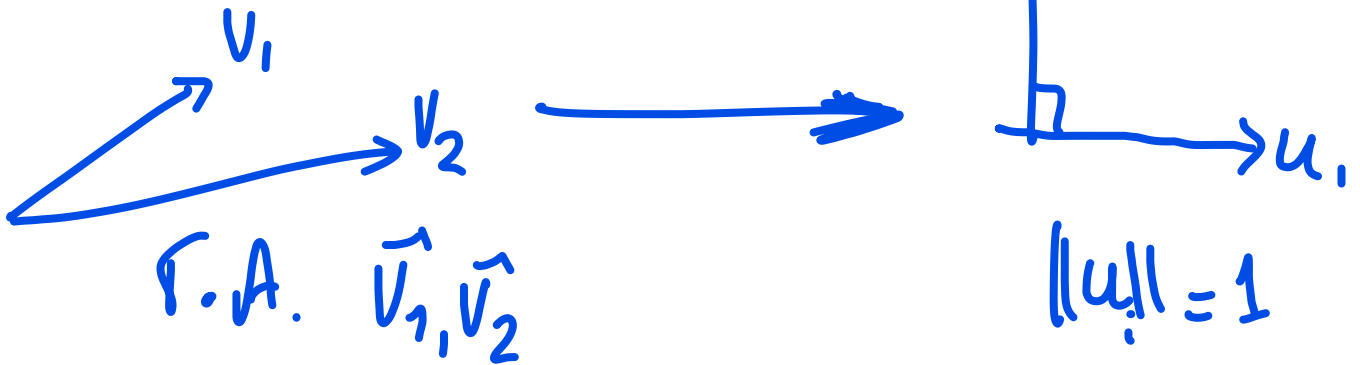


17 Μαρτίου :

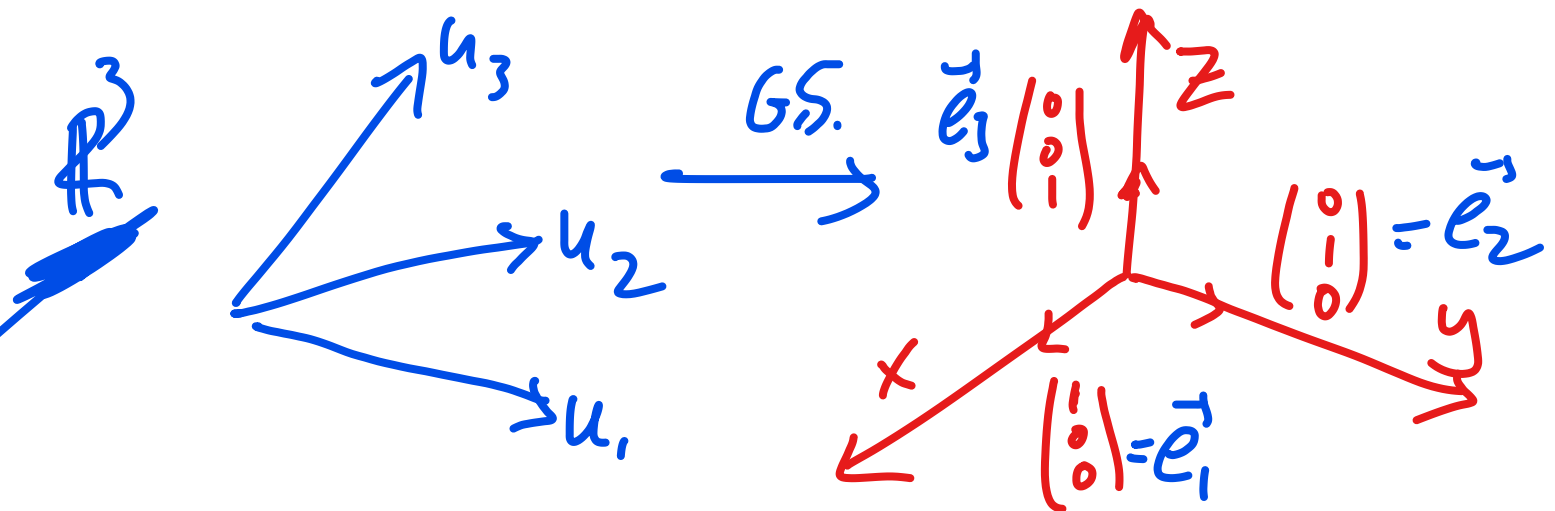
Gram-Schmidt

Ορθοκανονικοποίηση
G-S, QR



$\{v_i\}, \{u_i\} =$ βάσεις του
ιδίου χώρου

Βάση $v_i \xrightarrow{\text{G.S.}}$ Βάση $u_i = \text{ΟΚΒ}$



Ⓐ Έστω $\vec{u}_1 = \vec{v}_1$.

$\vec{v}_1 = \gamma \nu \sigma \tau \alpha$

Ⓑ $\vec{u}_2 = \vec{v}_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} \vec{u}_1$

Αφαιρούμε
τα u_1
από τα v_2 "μέχρι"

Ⓒ $\vec{u}_3 = \vec{v}_3 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} \vec{u}_1 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} \vec{u}_2$

ΚΑΤ. ΟΜΟΙΟΣ

Ⓓ ΚΑΝΟΜΙΚΟΠΟΙΗΣΗ ΤΩΝ \vec{u}_i

ex 1: Έστω \mathbb{R}^4 : $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ -3 \end{pmatrix}$
 $\vec{v}_3 = \begin{pmatrix} 0 \\ 3 \\ 6 \\ 5 \end{pmatrix}$ $\vec{v}_1, \vec{v}_2, \vec{v}_3 = \text{Γ.Α.}$

$\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \lambda_3 \vec{v}_3 = 0 \Rightarrow \lambda_i = 0$

$$[v_1, v_2, v_3] = \text{span} \text{ υποσφαι } \subseteq \mathbb{R}^4 = [u_1, u_2, u_3] \quad ?$$

G.S.

(A) Ορίζω $\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

(B) $\vec{u}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \cdot \vec{u}_1 =$

$$\begin{pmatrix} 2 \\ 2 \\ 0 \\ -3 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 2 \\ 2 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle}{\sqrt{1^2 + 0^2 + 1^2 + 0^2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \\ 0 \\ -3 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -3 \end{pmatrix}$$

$$= \vec{u}_2.$$

$$\textcircled{r} \vec{u}_3 = \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \cdot \vec{u}_1 - \frac{\langle \vec{v}_3, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} \cdot \vec{u}_2$$

$$= \begin{pmatrix} 0 \\ 3 \\ 6 \\ 5 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ 6 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{1^2+1^2}} \cdot \vec{u}_1 - \frac{\begin{pmatrix} 0 \\ 3 \\ 6 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \\ -3 \end{pmatrix}}{\sqrt{1+9+4+9}} \cdot \vec{u}_2$$

$$\begin{pmatrix} 0 \\ 3 \\ 6 \\ 5 \end{pmatrix} - \frac{6}{2} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{-15}{15} \begin{pmatrix} 1 \\ 2 \\ -1 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 3 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 2 \\ 2 \end{pmatrix}$$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -3 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} -2 \\ 5 \\ 2 \\ 2 \end{pmatrix}$$



Κανονικοποιώ:

$$\begin{pmatrix} -\frac{2}{\sqrt{37}} \\ \frac{5}{\sqrt{37}} \\ \frac{2}{\sqrt{37}} \\ \frac{2}{\sqrt{37}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{15}} \\ -\frac{1}{\sqrt{15}} \\ -\frac{3}{\sqrt{15}} \end{pmatrix}$$

~~OKB.~~

$n \times 2$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

~~G.S?~~

(A) $\vec{u}_1 = \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{array}{|l} \left(\begin{matrix} 1 \\ 2 \end{matrix} \right) \cdot \left(\begin{matrix} 3 \\ 4 \end{matrix} \right) = \\ 11 \end{array}$$

(B) $\vec{u}_2 = \vec{v}_2 - \frac{\langle \vec{u}_1, \vec{v}_2 \rangle}{\|\vec{u}_1\|^2} \cdot \vec{u}_1 =$

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{11}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} =$$

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 11/5 \\ 22/5 \end{pmatrix} = \begin{pmatrix} 4/5 \\ -2/5 \end{pmatrix}$$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\vec{u}_2 = \begin{pmatrix} 4/5 \\ -2/5 \end{pmatrix}$$

(r)

$$\vec{u}_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$$\|\vec{u}_2\| = \sqrt{\frac{20}{25}} = \frac{\sqrt{20}}{5}$$

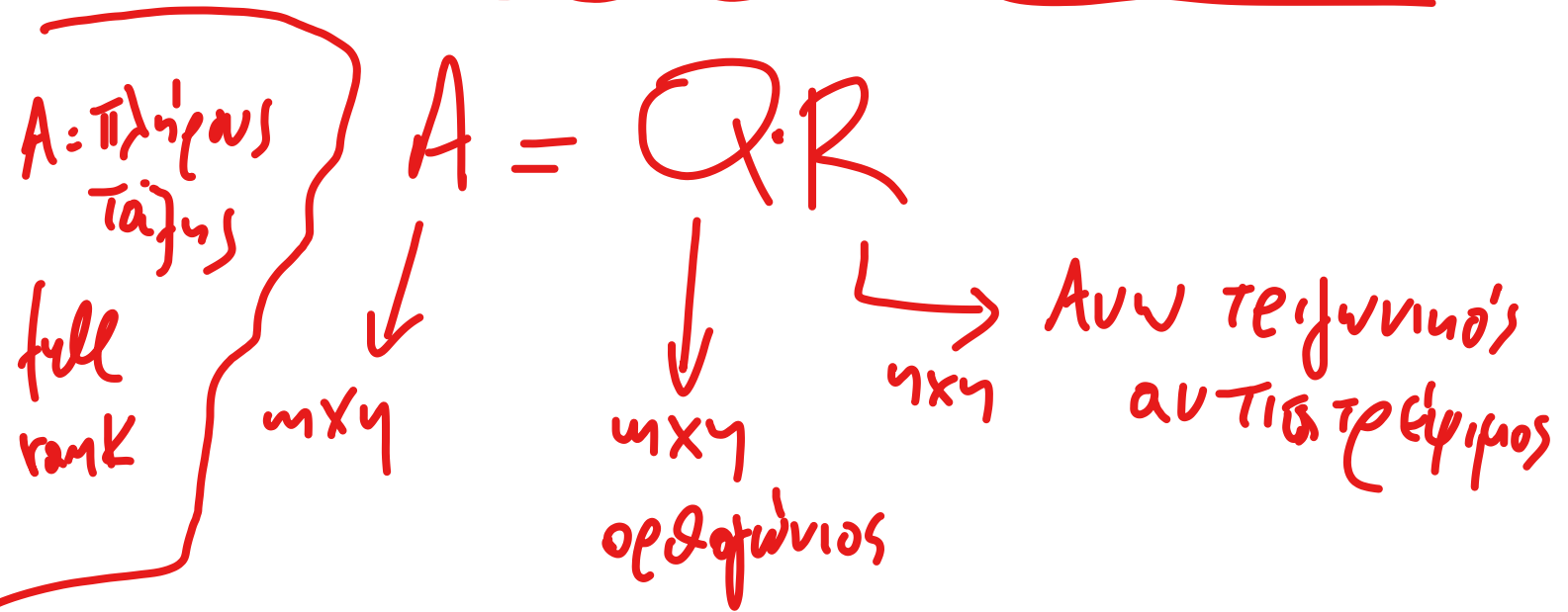
$$\vec{u}_2 = \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}$$

OKB

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

~~_____~~
 v_1, v_2

Η ΠΑΡΑΓΟΝΤΟΠΟΙΗΣΗ QR



$$Q^T Q = I$$

$24 = 6 \cdot 4$

$A = LDU = L D L^T =$
Cholesky: $L D^{1/2} D^{1/2} L^T =$
 $R \cdot R^T$

$A = A^T$

ΠΑΡΑΓΟΝΤΟΠΟΙΗΣΕΙΣ :

$$A = LU$$

Αναγωγή Gauss

$$A = QR$$

Κανονικοποίηση G-S

$A: m \times n$, αντιστρέφ. $\left| \right. A = m \times n$, full rank
(full rank)

(ΜΟΝΑΔΙΚΕΣ)

QR $A \rightarrow$ G.S. (συνήθεις)

\rightarrow νέες συνήθεις = Q ορθογώνιος

$$A = QR \Rightarrow Q^T A = Q^T Q R \Rightarrow$$

$$\underline{R = Q^T A.}$$

πχ: $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ QR?

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \xrightarrow{\text{G.S.}} (n \times 2) \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}$$

~~OR?~~

$$R = Q^T A =$$

$$\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \sqrt{5} & \frac{19}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{bmatrix} \quad R$$

(R: Ορθών Διαγώνιος)

Βιοζήτησις - Χρησιμοποίησις:

$$A = Q \cdot R$$

1) Σύστημα $Ax = b \Rightarrow$

$$QRx = b \Rightarrow \underline{Rx = Q^T b}$$

Τετραγωνικό σύστημα \Rightarrow λύσιμο

2) Λ.Ε.Τ. της $Ax = b$

$$\Rightarrow A^T Ax = A^T b$$

\Rightarrow ίδια λύση με $Rx = Q^T b$

Απόδ.: Έστω $A = QR \Rightarrow A^T = R^T Q^T$

Κανονική
Εξίσωση:

$$A^T Ax = A^T b \Rightarrow R^T Q^T Q R x = R^T Q^T b$$

$$\Rightarrow R^T \cdot Rx = R^T \cdot Q^T \cdot b \quad \cdot (R^T)^{-1}$$

$$\Rightarrow \underline{Rx = Q^T b}$$

Συμπερασματικά
 $A : \text{r.a.}$

$A \neq 0 \quad A = \text{full rank}$