ΦΡΟΝΤΙΣΤΗΡΙΟ

9 Ιουνίου

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1. QR ?

$$A = \begin{array}{cc} \vec{x}_1 & \vec{x}_2 \\ \end{array}$$
$$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\underset{m\times n}{A} = \overset{m\times n}{Q}\,\overset{n\times n}{R}$$

Q → ορθογώνιος

R → άνω τριγωνικής

$$Q^TQ = I$$

$$\left(A = QR \Rightarrow Q^TA = R\right)$$

Με G-S ορθοκανονικοποίηση Q

Θα βρω $\vec{v}_1, \vec{v}_2$: $\left\{ \begin{array}{l} \vec{v}_1 \perp \vec{v}_2 \\ \|v_i\| = 1 \end{array} \right.$

$\vec{x}_1 = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \quad \vec{x}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

A Θέτω $\vec{v}_1 = \vec{x}_1 = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$

B $v_2 = \vec{x}_2 - \dfrac{\langle x_1, x_2 \rangle}{\|x_1\|^2} \cdot \vec{x}_1$

$= \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \dfrac{15}{45} \cdot \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \longrightarrow v_2$

$\vec{v}_1 = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ KANONIK.

$$\vec{v}_1 = \begin{pmatrix} 3/\sqrt{45} \\ 6/\sqrt{45} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\cancel{3}\,1}{\cancel{3}\sqrt{5}} \\ \frac{\cancel{6}\,2}{\cancel{3}\sqrt{5}} \\ 0 \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{R = Q^T \cdot A} \Longrightarrow$$

$$R = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{15}{\sqrt{5}} & \frac{5}{\sqrt{5}} \\ 0 & 2 \end{bmatrix} = R$$

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② ΛΕΤ και ο A^+ ?

της

$$\boxed{(A^T A)^{-1} \cdot A^T = A^+} \qquad Ax = b:$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

Λυση:

$$Ax = b \Rightarrow A^T A x = A^T b$$

(ΚΑΝΟΝΙΚΗ ΕΞΙΣΩΣΗ) πάντα έχει λύση

$$x = \underbrace{(A^TA)^{-1} \cdot A^T}\cdot b \quad (x = A^+ b)$$

- $A^TA = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 24 \end{bmatrix}$

$$(A^TA)^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{24} \end{bmatrix}$$

- $(A^TA)^{-1} \cdot A^T = \begin{bmatrix} 1/3 & 0 \\ 0 & \frac{1}{24} \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1/3 & -1/3 & \frac{1}{3} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \end{bmatrix} = A^+$$

- $\hat{x} = A^+ \cdot b$

$$\hat{x} = \begin{bmatrix} +\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix}$$

λυση
του $Ax = b$

$$\hat{x} = \begin{pmatrix} 3 \\ 0,5 \end{pmatrix}$$

$A\hat{x} \neq b$?

$$A\hat{x} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 0,5 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix} = b_1$$

$b_1 = P_A b$

$\boxed{\text{σφάλμα}}$

σφάλμα $= \| b - b_1 \| =$

$$\left\| \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\|$$

$$= \sqrt{1^2 + 1^2} = \underline{\underline{\sqrt{2}}}$$

νόρμα σφάλματος $= 1{,}41$

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3 ΔΙΑΓΩΝΟΠΟΙΗΣΗ

i) $A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$ ιδιοτιμές / ιδιοδιαν.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(5-\lambda) + 1 = 0 \Rightarrow$$

$$\lambda^2 - 8\lambda + 16 = 0 \Rightarrow \lambda = 4$$

διπλή.

ΙΔΙΟΔΙΑΝ.?

$$A\vec{u} = 4\vec{u}$$

$$\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 4\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow$$

$$\begin{cases} 3x - y = 4x \\ x + 5y = 4y \end{cases} \rightarrow \begin{cases} x = -y \\ x = -y \end{cases}$$

$$\vec{u} = \begin{pmatrix} x \\ -x \end{pmatrix} \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Διαγων ?

$\boxed{\text{ΟΧΙ}}$

ΛΕΙΠΕΙ ΕΝΑ ΙΔΙΟΔ.

τοτε: JORDAN $\boxed{A = P J \cdot P^{-1}}$

χαρ/κο: $(x-4)^2 = 0$

ελαχιστο; $\begin{cases} x-4 & ή \\ (x-4)^2 \end{cases}$

$A - 4I = 0$? $\boxed{\text{Οχι το } x-4}$

Αρα ελαχιστο $= (x-4)^2$

$$J = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

iii) $B = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

διαγωνοπ.?

$\lambda = 1, 2, 3 \longrightarrow$ έχει 3 ιδιοτιμές.

$\Rightarrow$ ΝΑΙ.

(A) για $\lambda = 2$? Ιδιοδιάν.? $B\vec{u} = 2\vec{u}$

$$\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{cases} -x + 4y - 2z = 2x \\ -3x + 4y = 2y \\ -3x + y + 3z = 2z \end{cases} \longrightarrow$$

$$\begin{cases} -3x + 4y - 2z = 0 \\ -3x = -2y \\ -3x + y + z = 0 \end{cases} \longrightarrow y = \frac{3}{2}x$$

$$\begin{cases} -3x + 6x = 2z \\ -3x + 1{,}5x = -z \end{cases} \rightarrow$$

$$3x = 2z \rightarrow z = \frac{3}{2}x$$

$$\cancel{-1{,}5x = -z}$$

$$\vec{u} = \begin{pmatrix} x \\ \frac{3}{2}x \\ \frac{3}{2}x \end{pmatrix} \xrightarrow{nx} \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

B $\lambda = 1$: $B\vec{v} = 1 \cdot \vec{v} \Rightarrow$

$$\begin{cases} -x + 4y - 2z = x \\ -3x + 4y = y \\ -3x + y + 3z = z \end{cases} \longrightarrow \boxed{x = y}$$

ΑΝΤΙΚΑΘΙΣΤΩ $\Longrightarrow \begin{cases} \boxed{x = z} \\ -2x = -2z \end{cases}$ ✓

$$\vec{v} = \begin{pmatrix} x \\ x \\ x \end{pmatrix} \xrightarrow{\pi x} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Γ) Για την $\lambda = 3$: $B\vec{w} = 3\vec{w}$

$$\begin{cases} -x + 4y - 2z = 3x \\ -3x + 4y = 3y \rightarrow y = 3x \\ -3x + y + \cancel{3z} = \cancel{3z} \longrightarrow y = 3x \end{cases}$$

Άρα $-4x + 12x - 2z = 0$

$$z = 4x$$

$$\vec{w} = \begin{pmatrix} x \\ 3x \\ 4x \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

Άρα

$$\boxed{B = P \Delta P^{-1}}$$

$$\Delta = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$P =$ αντίστοιχα ιδιοδιαν. σε στήλες

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix}$$

ii) $\underline{B^4 = ?}$

$$B^4 = (P\Delta P^{-1})^4 =$$

$$P\Delta^4 P^{-1} =$$

$$P\begin{bmatrix} 3^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 1^4 \end{bmatrix}P^{-1}.$$

$A\ (3\times 3)$ χαρ/κο :

4

$$x^3 - 2x^2 + 3x - 1 = 0$$

A^{-1}? συναρτήσει των A, I

Απο Θ.C-H: $A^3 - 2A^2 + 3A - I = 0$

$$\Rightarrow A^3 - 2A^2 + 3A = I \Rightarrow$$

$$A\underbrace{\left[A^2 - 2A + 3I\right]}_{A^{-1}} = I$$

γιατι: $\boxed{A \cdot A^{-1} = I}$

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(5) SVD $\longrightarrow \Sigma$?

$A = U\Sigma V^T$

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

3x4

$$AA^T = \dots \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

ιδιοτιμές του:
$$\begin{vmatrix} 3-\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\to \dots \lambda = 0,\ 3 \pm \sqrt{3}$$

$$\sigma = \sqrt{3+\sqrt{3}},\ \sqrt{3-\sqrt{3}},\ 0$$

$$\Sigma = \begin{bmatrix} \sqrt{3+\sqrt{3}} & 0 & 0 & 0 \\ 0 & \sqrt{3-\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$