The Binomial Theorem

Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$. We *define*

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} \quad \text{for } n = 1, 2, \ldots, \text{ and } \binom{\alpha}{0} := 1.$$

Since

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) = \alpha(\alpha-1)\Gamma(\alpha-1) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\Gamma(\alpha+1-n)$$

we can also write

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\Gamma(\alpha+1-n)}{\Gamma(\alpha+1-n)n!} = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha+1-n)}.$$

The Binomial Theorem of Newton states that,

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \qquad \text{for } |x| < 1. \tag{1}$$

Note that when α is a positive integer then the above series terminates after a finite number of terms. To give an example for the case where α is not a positive integer, suppose $\alpha = -r$, $k \in \mathbb{N}$. Then

$$\binom{-k}{n} = \frac{(-k)(-k-1)(-k-2)\cdots(-k-n+1)}{n!} = (-1)^n \frac{k(k+1)\cdots(n+k-1)}{n!} = \binom{n+k-1}{k-1}$$

and (1) gives

$$\frac{1}{(1+x)^{-k}} = \sum_{n=0}^{\infty} \binom{-k}{n} x^n = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} (-1)^n x^n.$$

As a second example consider the case $\alpha = -1/2$.

$$\begin{aligned}\binom{-1/2}{n} &= \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-n+1)}{n!} = (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!} \\ &= (-1)^n \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-1)\cdot(2n)}{2^n\, n!\cdot 2\cdot 4\cdot 6\cdots 2n} = (-1)^n \frac{(2n)!}{2^{2n}\, n! n!} = \frac{(-1)^n}{4^n}\binom{2n}{n}.\end{aligned} \tag{2}$$

Hence

$$\frac{1}{(1+x)^{\frac{1}{2}}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n = \sum_{n=0}^{\infty} \binom{2n}{n} (-1)^n \left(\frac{x}{4}\right)^n. \tag{3}$$

1 Problems

1. a) The negative binomial distribution is the distribution

$$p_n := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!} p^\alpha q^n, \qquad \alpha > 0, \quad n = 0, 1, 2, \ldots$$

where $p, q > 0$ and $p + q = 1$. Use the binomial theorem to show that its probability generating function is $\sum_{n=0}^{\infty} p_n z^n = \frac{p^\alpha}{(1-qz)^\alpha}$.

b) What is the probability generating function of the Poisson distribution given by $p_n = \frac{\lambda^n}{n!} e^{-\lambda}$, $n =$

$0, 1, 2, \ldots$?
c) Suppose now that in the Poisson distribution above the mean λ is itself a random variable with Gamma distribution given by $\beta \frac{(\lambda\beta)^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta\lambda}$, $\alpha, \beta > 0$. What is the resulting distribution? (Use the generating function obtained in part b.)

2. Suppose that $\{X_i\}$ is an i.i.d. sequence of exponential random variables with rate 1. Also $\{\xi_i\}$, $i = 1, 2, \ldots$, is an i.i.d. sequence (and also independent of $\{X_i\}$). The ξ_i take only two values, +1 and -1 with $\mathbb{P}(\xi_1 = +1) = p$, $\mathbb{P}(\xi_1 = -1) = q = 1 - p$. Finally let N is a geometric random variable with $\mathbb{P}(N = n) = (1 - \alpha)\alpha^{n-1}$, $n = 1, 2, \ldots$, and $\alpha \in (0, 1)$, independent of all the other random variables. Let $Y = \sum_{i=1}^{N} \xi_i \cdot X_i$. Find the moment generating function $M(\theta) = \mathbb{E}[e^{\theta Y}]$.