1 The Beta distribution

Consider two independent Gamma distributed random variables, X and Y, with shape parameters α and β respectively and scale parameter equal to 1. Thus $f_X(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x}$, $f_Y(y) = \frac{y^{\beta-1}}{\Gamma(\beta)} e^{-y}$, $x, y > 0$. Let

$$\begin{array}{rcl} U &=& \frac{X}{X+Y} \\ V &=& X+Y \end{array}, \qquad \begin{array}{rcl} X &=& UV \\ Y &=& V(1-U) \end{array}.$$

Also,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v(1-u) + uv = v.$$

The joint distribution of U, V, say $g(u,v)$, is given by

$$\begin{aligned} g(u,v) &= f_X(x(u,v)) f_Y(y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \\ &= \frac{(uv)^{\alpha-1}}{\Gamma(\alpha)} e^{-uv} \, \frac{(v(1-u))^{\beta-1}}{\Gamma(\beta)} e^{-v+vu} \, v \, \mathbf{1}(0<u<1) \, \mathbf{1}(v>0) \\ &= \frac{(uv)^{\alpha-1}}{\Gamma(\alpha)} e^{-uv} \, \frac{(v(1-u))^{\beta-1}}{\Gamma(\beta)} e^{-v+vu} \, v \, \mathbf{1}(0<u<1) \, \mathbf{1}(v>0) \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1} \mathbf{1}(0<u<1) \, \frac{v^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} e^{-v} \mathbf{1}(v>0). \end{aligned}$$

The marginal density of U is then

$$g_U(u) = \int_{v=0}^{\infty} g(u,v) dv = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1} \mathbf{1}(0<u<1). \tag{1}$$

Since $\int_0^1 g_U(u) du$ *must* be equal to 1 we conclude that

$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1} du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} =: B(\alpha,\beta). \tag{2}$$

Equation (1) gives the density of the Beta distribution.

2 A multidimensional extension – Dirichlet distribution

Consider now n independent, Gamma distributed random variables, X_i with shape parameters α_i, $i = 1, 2, \ldots, n$, and scale parameters all equal to 1. Let

$$\begin{array}{rcl} U_1 &=& \frac{X_1}{X_1+\cdots+X_n} \\ U_2 &=& \frac{X_1}{X_1+\cdots+X_n} \\ \vdots && \\ U_{n-1} &=& \frac{X_{n-1}}{X_1+\cdots+X_n} \\ U_n &=& X_1+\cdots+X_n \end{array}, \qquad \begin{array}{rcl} X_1 &=& U_1 U_n \\ X_2 &=& U_2 U_n \\ \vdots && \\ X_{n-1} &=& U_{n-1} U_n \\ X_n &=& U_n(1-U_1+\cdots+U_{n-1}) \end{array}.$$

The Jacobian of the transformation is

$$\frac{\partial(x_1,\ldots,x_n)}{\partial(u_1,\ldots,u_n)} = \begin{vmatrix} u_n & & & & u_1 \\ & u_n & & & u_2 \\ & & \ddots & & \\ & & & u_n & u_{n-1} \\ -u_n & -u_n & \cdots & -u_n & 1-u_1-\cdots-u_{n-1} \end{vmatrix} = u_n^{n-1}.$$

(The determinant is computed easily by adding the first $n-1$ rows to the last one.) The joint density of $U_1, \ldots, U_n$ is then given by

$$\begin{aligned} g(u_1,\ldots,u_n) &= \left(\prod_{i=1}^{n-1} \frac{(u_i u_n)^{\alpha_i - 1}}{\Gamma(\alpha_i)}\, e^{-u_i u_n}\right) \frac{(u_n(1-u_1-\cdots-u_{n-1}))^{\alpha_n-1}}{\Gamma(\alpha_n)}\, e^{-u_n(1-u_1-\cdots-u_{n-1})}\, u_n^{n-1} \\ &\quad \mathbf{1}(u_1+\cdots+u_{n-1}<1) \\ &= \left(\prod_{i=1}^{n-1} \frac{u_i^{\alpha_i-1}}{\Gamma(\alpha_i)}\right) \frac{(1-u_1-\cdots-u_{n-1})^{\alpha_n-1}}{\Gamma(\alpha_n)}\, \Gamma(\alpha_1+\cdots+\alpha_n)\, \mathbf{1}(u_1+\cdots+u_{n-1}<1) \\ &\quad \times \frac{u_n^{\alpha_1+\cdots+\alpha_n-1}}{\Gamma(\alpha_1+\cdots+\alpha_n)} e^{-u_n}. \end{aligned}$$

U_n is of course Gamma distributed and integrating out u_n in the above we obtain the joint density of the r.v.'s $U_1, \ldots, U_{n-1}$:

$$g(u_1,\ldots,u_{n-1}) = \left(\prod_{i=1}^{n-1} \frac{u_i^{\alpha_i-1}}{\Gamma(\alpha_i)}\right) \frac{(1-u_1-\cdots-u_{n-1})^{\alpha_n-1}}{\Gamma(\alpha_n)}\, \Gamma(\alpha_1+\cdots+\alpha_n)\, \mathbf{1}(u_1+\cdots+u_{n-1}<1). \tag{3}$$

This is the *Dirichlet* density on the set $\{(u_1,\ldots,u_{n-1}), u_i > 0, \sum_{i=1}^{n-1} u_i < 1\}$. It is a multidimensional generalization of the Beta density.

3 Probelms

1. *Dirichlet Distribution.* This problem refers to the section on the Dirichlet distribution above.

a) Show that the marginal distribution of U_1 is Beta with parameters $(\alpha_1, \sum_{i=2}^{n} \alpha_i)$. (Hint: This is not as hard as it may look at first sight. You may want to work out first the case $n = 3$, which means that there are two variables, u_1, u_2.)

b) What is the *conditional* density of $U_1, \ldots, U_{n-2}$ given that $U_{n-1} = x \in (0,1)$.

c) What is the conditional expectation $\mathbb{E}[U_i | U_{n-1} = x]$, $i = 1, \ldots, n-2$?

2. *The inverted Beta distribution* (some times also called the Beta prime).
a) Let X be a Beta(α,β) random variable and $U = X/(1-X)$. Find the density of X. This is called the inverted Beta.
b) Let X be a Gamma random variable with density

$$f(x|\lambda) = \lambda \frac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}\, e^{-\lambda x} \tag{4}$$

where $\alpha > 0$. In the above, λ is considered a fixed positive number. Now randomize λ, i.e. assume that it is itself random variable with density

$$f_\Lambda(\lambda) = \mu \frac{(\mu\lambda)^{\beta-1}}{\Gamma(\beta)}\, e^{-\mu\lambda} \tag{5}$$

where $\mu, \beta > 0$. Find the density of the mixture with respect to λ i.e. $f(x) = \int_0^\infty f(x|\lambda) f_\Lambda(\lambda) d\lambda$. Compare with your answer in question (a).

3. *Beta – Binomial*.

a) Suppose that the probability of success p in a Binomial random variable (n, p) is Beta distributed with parameters (α, β). Let X be the probability of k successes. Find $\mathbb{P}(X = k)$.

b) The Polya-Eggenberger Urn. Consider an urn initially containing α red and β black balls. We take a ball at random from the urn, noting the color and replacing the ball back in the urn, *together with one additional ball of the same color*. What is the probability that in the first n draws we observe k red balls. Compare the result with that obtained in question *(a)*.

Hint: Though it may be slightly surprising, the order in which we draw the balls does not matter. For instance, convince your selves that the probability of the events RBBRB and BBRBR is the same.