## 1 The Gamma function and the Gamma density

The Gamma function is defined for all $x > 0$ via the following integral:

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt. \tag{1}$$

Provided that $x > 1$ we can use integration by parts to prove the fundamental relationship

$$\Gamma(x) = (x-1)\Gamma(x-1). \tag{2}$$

Indeed,

$$\Gamma(x) \;=\; -\int_0^\infty t^{x-1} de^{-t} \;=\; t^{x-1}e^{-t}\big|_{t=0}^{\infty} + \int_0^\infty e^{-t}(x-1)t^{x-2}dt \;=\; (x-1)\Gamma(x-1).$$

In particular, from (2) and the fact that $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$, it follows that if $x$ is a natural number, say $n$, then $\Gamma(n) = (n-1)!$.

A continuous random variable with values on $[0, \infty)$ is Gamma-distributed with shape parameter $\alpha > 0$ and rate parameter $\lambda > 0$ when its density has the form

$$f(x) \;=\; \lambda \frac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x}, \qquad x > 0. \tag{3}$$

## 2 Convolution

Let $X, Y$, be *independent* random variables with densities $f_X(x)$ and $f_Y(y)$ respectively. Thus, the joint density of $X$ and $Y$ is $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. We wish to determine the density of the random variable $V := X + Y$. One possible way to do this is to define the transformation

$$\begin{aligned} U &= X, \\ V &= X + Y. \end{aligned}$$

Expressing the variables $x, y$ in terms of the variables $u, v$ we have

$$\begin{aligned} x &= u, \\ y &= v - u. \end{aligned}$$

The Jacobian determinant of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.$$

Then, the joint distribution of $U, V$ is given by

$$f_{U,V}(u,v) \;=\; f_{X,Y}(u, v-u) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \;:=\; f_X(u)f_Y(v-u). \tag{4}$$

Therefore, it is enough to determine the *marginal density* $f_V(v) := \int_{-\infty}^\infty f_{U,V}(u,v)dv$. Hence,

$$f_V(v) \;=\; \int_{-\infty}^\infty f_X(u)f_Y(v-u)du. \tag{5}$$

The above integral is called the *convolution* of the densities $f_X$, $f_Y$,

## 3 Problems

Read Chapter 1 of Gut, *An Intermediate Course in Probability Theory*.

**1.** Let $X$ be a standard normal random variable and $Y = e^X$. Find the density of $Y$.

**2.** Let $X$ be uniformly distributed on $[0, 1]$ and $Y = \frac{2X}{1+X}$. Find the density of $Y$.

**3.** Let $X$, $Y$ be uniformly distributed on $[0, 1]$ and independent. Find the density of $V = XY$.

**4.** Solve problems 39 and 41 on page 28, Chapter 1, of Gut.