Bijective Reparameterization Covariance of OE Estimators

Suppose that the researcher is interested in estimating $\varphi(\theta_0)$, with $\varphi: \Theta \to \Phi$, Φ lies also in some Euclidean space, and the reparameterization function φ is bijective (this essentially means that φ "simply renames" Θ). Given the OE θ_n for θ_0, is the OE for $\varphi_0 := \varphi(\theta_0)$, $\varphi(\theta_n)$. This would imply that the OE transforms covariantly w.r.t. bijections, i.e.

$$OE(\varphi(\theta_0)) = \varphi(OE(\theta_0)).$$

Disregarding without loss of generality optimization errors, we know that

$$C_n(\theta_n) = \sup_{\theta \in \Theta} C_n(\theta) \quad (*)$$

Also, for φ^{-1} the inverse function, well defined due to bijectivity, we have that

$$\Theta = \varphi^{-1}(\Phi) = \{ \theta \in \Theta : \theta = \varphi^{-1}(\phi), \exists \phi \in \Phi \},$$

hence:

$$\sup_{\theta \in \Theta} C_n(\theta) = \sup_{\theta \in \varphi^{-1}(\Phi)} C_n(\theta) = \sup_{\varphi \in \Phi} C_n(\varphi^{-1}(\phi))$$

$= C_n(\varphi^{-1}(\phi_n))$ where ϕ_n is the OE for

$(C_n \circ \varphi^{-1})(\varphi)$, i.e. the OE for $\varphi(\theta_0)$.

Thereby, $C_n(\Theta_n) = C_n(\varphi^{-1}(\phi_n))$ and even if optimizers are non unique, it holds $\Theta_n = \varphi^{-1}(\varphi_n)$ $\Leftrightarrow$ $\varphi_n = \varphi(\Theta_n)$ establishing covariance.

E.g. In the context of the LS model

Suppose that $\varphi(\theta) = EXP(\theta) := \begin{pmatrix} \exp(\theta_1) \\ \vdots \\ \exp(\theta_k) \end{pmatrix}$.

Then the OLSE for $\varphi(\theta_0)$ is

$$EXP((X_n'X_n)^{-1}X_n'Y_n).$$ (why is EXP, 1-1?) $\boxtimes$

When φ is not bijective, then covariance will generally fail.

What then about the limit theory of $\varphi(\theta_n)$?

if θ_n is (weakly) consistent, and φ is moreover continuous, then due to the CMT

$\varphi(\theta_n) \xrightarrow{P} \varphi(\theta_0)$ establishing consistency for φ_n.

If $r_n(\theta_n - \theta_0) \rightsquigarrow Z_{\theta_0}$, and φ is continuously differentiable in some B_{θ_0}, then the Delta Method implies that $r_n(\varphi(\theta_n) - \varphi(\theta_0))$

$$\rightsquigarrow \frac{\partial \varphi}{\partial \theta}(\theta_0) Z_{\theta_0} \implies r_n(\varphi_n - \varphi_0) \rightsquigarrow \frac{\partial \varphi}{\partial \theta'}(\theta_0) Z_{\theta_0}$$

[this can be extended to the case where $\theta_0 \in \mathcal{B}_d\Theta$ via the use of one-sided derivatives]

E.g. under the assumptions used in the OLSE case and since $\dfrac{\partial EXP(\theta)}{\partial \theta'} = \begin{pmatrix} \exp(\theta_1) & 0 & \cdots & 0 \\ 0 & \exp(\theta_2) & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \exp(\theta_k) \end{pmatrix}$

and since $n^{1/2}(\theta_n - \theta_0) \rightsquigarrow z_{\theta_0} \sim N(0_{k\times 1}, U_{x'x})$

then $n^{1/2}(\varphi_n - \varphi_0) \rightsquigarrow N(0_{k\times 1}, V)$ with

$$V = \begin{pmatrix} \exp(\theta_1) & 0 & \cdots & 0 \\ 0 & \exp(\theta_2) & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \exp(\theta_k) \end{pmatrix} U_{x'x}^{-1} \begin{pmatrix} \exp(\theta_1) & 0 & \cdots & 0 \\ 0 & \exp(\theta_2) & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \exp(\theta_k) \end{pmatrix}$$

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