Econometrics 2

Lecture 15: Asymptotic Normality of the Instrumental Variables Estimator

AUEB | Spring Semester 2026

Taught by: Prof. S. Arvanitis

Notes transcribed & digitized by: T. Kourtalis & Ch. Dourgoutis

1. Setup: Score and Hessian of the IV Objective Function

Recall from Lecture 14 that for the IV objective function with weighting matrix ω:

$$\frac{\partial M_n(\beta)}{\partial \beta} = -2\,\frac{X_n' Z_n}{n}\,\omega\,\frac{Z_n'(Y_n - X_n\beta)}{n}$$

$$\frac{\partial^2 M_n(\beta)}{\partial \beta\,\partial \beta'} = 2\,\frac{X_n' Z_n}{n}\,\omega\,\frac{Z_n' X_n}{n}$$

Evaluating the scaled score at β_0:

$$\sqrt{n}\,\frac{\partial M_n(\beta_0)}{\partial \beta} = -2\,\frac{X_n' Z_n}{n}\,\omega\,\frac{Z_n'\varepsilon_n}{\sqrt{n}}$$

$$= -2\left(\frac{1}{n}\sum_{i=1}^{n} X_{(i)}' Z_{(i)}\right)\omega\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_{(i)}\varepsilon_{(i)}\right) \qquad (*)$$

2. Asymptotic Behavior of $(*)$

Regarding the asymptotic behavior of $(*)$, we assume for reasons of simplicity that $(X_{(i)}, Z_{(i)}, \varepsilon_{(i)})$ are i.i.d. with respect to i.

LLN Term

Since $\mathbb{E}(Z_{(i)}' X_{(i)})$ exists, then because of Kolmogorov's Law of Large Numbers:

$$\frac{1}{n}\sum_{i=1}^{n} X_{(i)}' Z_{(i)} \xrightarrow{p} \mathbb{E}(X_{(i)}' Z_{(i)}) = Q_{Z'X}'$$

where from the derivation of consistency we had that $Q_{Z'X}'$ has rank p.

CLT Term

If furthermore it holds that:

$$\mathbb{E}(Z'_{(i)}\varepsilon_{(i)}(Z'_{(i)}\varepsilon_{(i)})') = \mathbb{E}(Z'_{(i)}\varepsilon^2_{(i)}Z_{(i)}) := V_{Z'\varepsilon}$$

exists and is non-singular, then because of the Central Limit Theorem we have:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_{(i)}\varepsilon_{(i)} \xrightarrow{d} Z^* \sim \mathcal{N}(0_{q\times 1},\, V_{Z'\varepsilon})$$

Combining via Slutsky

Therefore, since:

$$\frac{1}{n}\sum_{i=1}^{n} X'_{(i)}Z_{(i)} \xrightarrow{p} Q'_{Z'X} \qquad \text{and} \qquad \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_{(i)}\varepsilon_{(i)} \xrightarrow{d} Z^*$$

we consequently have:

$$(*) \xrightarrow{d} -2\,Q'_{Z'X}\,\omega\,Z^* \sim \mathcal{N}(0_{p\times 1},\; 4\,Q'_{Z'X}\,\omega\,V_{Z'\varepsilon}\,\omega\,Q_{Z'X})$$

3. Hessian Convergence

Finally we deal with the asymptotic behavior of the Hessian at the intermediate point β_n^* from the Taylor expansion:

$$\frac{\partial^2 M_n(\beta_n^*)}{\partial\beta\,\partial\beta'} = 2\,\frac{X'_n Z_n}{n}\,\omega\,\frac{Z'_n X_n}{n} \xrightarrow{p} 2\,Q'_{Z'X}\,\omega\,Q_{Z'X}$$

where:

$$\frac{1}{n}\sum_{i=1}^{n} X'_{(i)}Z_{(i)} \xrightarrow{p} \mathbb{E}(X'_{(i)}Z_{(i)}) = Q'_{Z'X} \qquad \text{and} \qquad \frac{1}{n}\sum_{i=1}^{n} Z'_{(i)}X_{(i)} \xrightarrow{p} \mathbb{E}(Z'_{(i)}X_{(i)}) = Q_{Z'X}$$

This limit is positive definite when ω is positive definite and $\text{rank}(Q_{Z'X}) = p$. We therefore set:

$$2\,Q'_{Z'X}\,\omega\,Q_{Z'X} := H(\beta_0)$$

4. Final Result

Consequently, and since:

0. $\beta_0 \in Int(\Theta)$
1. $(X_{(i)}, Z_{(i)}, \varepsilon_{(i)})$ are i.i.d. with respect to i

2. $Q_{Z'X} := \mathbb{E}(Z'_{(i)}X_{(i)})$ exists and has rank p

3. $V_{Z'\varepsilon} := \mathbb{E}(Z'_{(i)}\varepsilon^2_{(i)}Z_{(i)})$ exists and is invertible

then:

$$\sqrt{n}(b_n - \beta_0) \xrightarrow{d} \mathcal{N}\big(0_{p\times 1},\ (Q'_{Z'X}\,\omega\,Q_{Z'X})^{-1}\,Q'_{Z'X}\,\omega\,V_{Z'\varepsilon}\,\omega\,Q_{Z'X}\,(Q'_{Z'X}\,\omega\,Q_{Z'X})^{-1}\big)$$

5. Observations on the Asymptotic Variance

- The rate of convergence of β_n is $\sqrt{n}$ and is independent of ω.
- The asymptotic distribution is Gaussian (Normal), which is independent of ω.
- The Asymptotic Variance (AsyVar) depends on ω.

Question: Can the AsyVar be independent of ω?

When $p = q$ (Exact Identification):
The matrix $Q_{Z'X}$ becomes square $p \times p$.
The matrix ω becomes square $p \times p$.
The matrix $V_{Z'\varepsilon}$ becomes square $p \times p$.

And all of them are invertible, therefore:

$$\begin{aligned} AsyVar &= (Q'_{Z'X}\,\omega\,Q_{Z'X})^{-1}\,Q'_{Z'X}\,\omega\,V_{Z'\varepsilon}\,\omega\,Q_{Z'X}\,(Q'_{Z'X}\,\omega\,Q_{Z'X})^{-1} \\ &= Q^{-1}_{Z'X}\,\omega^{-1}\,(Q'_{Z'X})^{-1}\,Q'_{Z'X}\,\omega\,V_{Z'\varepsilon}\,\omega\,Q_{Z'X}\,Q^{-1}_{Z'X}\,\omega^{-1}\,(Q'_{Z'X})^{-1} \\ &= Q^{-1}_{Z'X}\,V_{Z'\varepsilon}\,(Q'_{Z'X})^{-1} \end{aligned}$$

When $p = q$ and $\beta_0 \in Int(\Theta)$ then:
The estimator becomes asymptotically independent of ω.
(This does not hold when β_0 is a boundary point.)

When $q > p$ (Over-Identification):
The asymptotic variance depends on ω. Therefore, the question of the optimal choice of ω arises in order to achieve the smallest possible AsyVar.