Econometrics 2

Lecture 14: Asymptotic Normality of the OLSE
& Instrumental Variables Estimator

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1. Applying the General Theory to the OLSE

We apply our general asymptotic theory to our examples (except for LAD). In each case, we examine conditions that imply Assumptions 1–5 of our general theory. We begin with the OLSE case.

Assuming initially that $\beta_0 \in Int(\Theta)$, we have that:

$$\frac{\partial \mu_n(\beta)}{\partial \beta} = -\frac{2X_n' X_n}{n}(\beta_0 - \beta) - \frac{2X_n' \varepsilon_n}{n}$$

$$\frac{\partial^2 \mu_n(\beta)}{\partial \beta \, \partial \beta'} = \frac{2X_n' X_n}{n}$$

With probability approaching 1:

$$\hat{\beta}_n - \beta_0 = -\left(\frac{\partial^2 \mu_n}{\partial \beta \, \partial \beta'}(\beta_n^*)\right)^{-1} \frac{\partial \mu_n(\beta_0)}{\partial \beta}$$

If additionally $(X_{(i)}, \varepsilon_{(i)})$ are i.i.d. with respect to i, and $\mathbb{E}(\varepsilon_{(i)}^2 X_{(i)}' X_{(i)})$ exists, then because the classical CLT is applicable we have that:

$$\sqrt{n}\,\frac{\partial \mu_n(\beta_0)}{\partial \beta} = -\frac{2X_n' \varepsilon_n}{\sqrt{n}} = -\frac{2}{\sqrt{n}} \sum_{i=1}^{n} X_{(i)}' \varepsilon_{(i)} \xrightarrow{d} Z \sim N\big(0_{p\times 1},\ 4\,\mathbb{E}\big(\varepsilon_{(i)}^2 X_{(i)}' X_{(i)}\big)\big)$$

From our general theory.

2. Hessian Convergence (Assumption 5)

Regarding Assumption 5 of our general theory: we remember that in the consistency study of the estimator, in the case where $(X_{(i)}, \varepsilon_{(i)})$ are i.i.d. and since $\mathbb{E}(X_{(i)}' X_{(i)})$ exists, Kolmogorov's Law of Large Numbers implies that:

$$\frac{1}{n} X_n' X_n = \frac{1}{n} \sum_{i=1}^{n} X_{(i)}' X_{(i)} \xrightarrow{p} \mathbb{E}(X_{(i)}' X_{(i)})$$

Therefore, due to the Continuous Mapping Theorem:

$$\frac{2}{n}X_n'X_n \xrightarrow{p} 2\,\mathbb{E}(X_{(i)}'X_{(i)})$$

and consequently, if we denote by β_n^* the intermediate point of the Mean Value Theorem of our theory:

$$\frac{\partial^2 \mu_n(\beta_n^*)}{\partial\beta\,\partial\beta'} \xrightarrow{p} 2\,\mathbb{E}(X_{(i)}'X_{(i)}) := H(\beta_0)$$

$H(\beta_0)$ does not depend on β_0.

Assumption 5 will be satisfied ultimately as long as $\operatorname{rank}\mathbb{E}(X_{(i)}'X_{(i)}) = p$.

3. The Asymptotic Distribution of the OLSE

Therefore, from the above, our general theory yields:

$$\sqrt{n}(\hat{\beta}_n - \beta_0) \xrightarrow{d} N\left(0_{p\times 1},\ \underbrace{\frac{1}{2}\mathbb{E}^{-1}(X_{(i)}'X_{(i)})}_{H^{-1}(\beta_0)}\cdot\underbrace{4\,\mathbb{E}(\varepsilon_{(i)}^2 X_{(i)}'X_{(i)})}_{V}\cdot\underbrace{\frac{1}{2}\mathbb{E}^{-1}(X_{(i)}'X_{(i)})}_{H^{-1}(\beta_0)}\right)$$

$$= N\big(0_{p\times 1},\ \mathbb{E}^{-1}(X_{(i)}'X_{(i)})\ \mathbb{E}(\varepsilon_{(i)}^2 X_{(i)}'X_{(i)})\ \mathbb{E}^{-1}(X_{(i)}'X_{(i)})\big)$$

Due to the above:

$$\left(\frac{1}{n}X_n'X_n\right)^{-1} \xrightarrow{P} \mathbb{E}^{-1}(X_{(i)}'X_{(i)})$$

so it will be a consistent estimator of it.

4. Application to the Instrumental Variables Estimator

Our general theory is directly applicable also to the version of the linear model with instrumental variables. Given the weighting matrix ω, the objective function is:

$$M_n(\beta) = \frac{1}{n^2}(Y_n - X_n\beta)'\,Z_n\,\omega\,Z_n'(Y_n - X_n\beta)$$

- We assume that $\beta_0 \in Int(\Theta)$.
- The differentiability conditions are satisfied since M_n is a quadratic function of β.

We have that:

$$\frac{\partial M_n(\beta)}{\partial\beta} = -2\,\frac{X_n'Z_n}{n}\,\omega\,\frac{Z_n'(Y_n - X_n\beta)}{n}$$

$$\frac{\partial^2 M_n(\beta)}{\partial\beta\,\partial\beta'} = 2\,\frac{X_n'Z_n}{n}\,\omega\,\frac{Z_n'X_n}{n}$$

Evaluating the scaled score at β_0:

$$\begin{aligned}
\sqrt{n}\,\frac{\partial M_n(\beta_0)}{\partial \beta} &= -2\,\frac{X_n' Z_n}{n}\,\omega\,\frac{Z_n'\varepsilon_n}{\sqrt{n}} \\
&= -2\left(\frac{1}{n}\sum_{i=1}^{n} X_{(i)}' Z_{(i)}\right)\omega\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_{(i)}\varepsilon_{(i)}\right) \qquad (*)
\end{aligned}$$

The asymptotic analysis of $(*)$ is carried out in Lecture 15.