Econometrics 2

Lecture 12: Asymptotic Identification

IV Estimator Consistency

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1. Asymptotic Weak Consistency of the IV Estimator

Let us examine the issue of weak consistency for the Instrumental Variables (IV) estimator within the framework of the linear model with instrumental variables.

The objective function for the IV estimator is:

$$M_n(\beta) = \frac{1}{n^2}(Y_n - X_n\beta)' Z_n W Z_n' (Y_n - X_n\beta)$$

Since the model is well-specified, can we design, based on our general theory, sufficient conditions for the weak consistency of β_n?

2. Expansion of the Objective Function

Because of the correct specification, we know $Y_n = X_n\beta_0 + \epsilon_n$. We substitute this into the objective function:

$$\begin{aligned} M_n(\beta) &= \frac{1}{n^2}(\epsilon_n + X_n\beta_0 - X_n\beta)' Z_n W Z_n' (\epsilon_n + X_n\beta_0 - X_n\beta) \\ &= \frac{1}{n^2}\Big(\epsilon_n + X_n(\beta_0 - \beta)\Big)' Z_n W Z_n' \Big(\epsilon_n + X_n(\beta_0 - \beta)\Big) \\ &= \frac{1}{n^2}\Big(\epsilon_n' Z_n W Z_n' \epsilon_n + (\beta_0 - \beta)' X_n' Z_n W Z_n' X_n (\beta_0 - \beta) + 2(\beta_0 - \beta)' X_n' Z_n W Z_n' \epsilon_n\Big) \end{aligned}$$

By distributing the $\frac{1}{n^2}$ term (placing a $\frac{1}{n}$ under each matrix), we can rewrite this as:

$$M_n(\beta) = \left(\frac{\epsilon_n' Z_n}{n}\right) W \left(\frac{Z_n' \epsilon_n}{n}\right) + (\beta_0 - \beta)' \left(\frac{X_n' Z_n}{n}\right) W \left(\frac{Z_n' X_n}{n}\right)(\beta_0 - \beta) + 2(\beta_0 - \beta)' \left(\frac{X_n' Z_n}{n}\right) W \left(\frac{Z_n' \epsilon_n}{n}\right)$$

3. Asymptotic Behavior of the IV Components

If we know how one component behaves, we also know how its transpose behaves due to the Continuous Mapping Theorem (or continuous transformation properties). Let us establish our core assumptions:

a) Convergence of the Instrument-Regressor Covariance

We assume that:

$$\frac{Z_n' X_n}{n} \xrightarrow{P} Q_{Z'X}$$

where $Q_{Z'X}$ is a non-stochastic $q \times p$ matrix, and we assume $rank(Q_{Z'X}) = p$.

For example, if the pairs $(Z_{(i)}, X_{(i)})$ are iid with respect to i and the expectation $\mathbb{E}(Z_{(1)}' X_{(1)})$ exists, then due to Kolmogorov's Law of Large Numbers (LLN):

$$Q_{Z'X} = \mathbb{E}(Z_{(1)}' X_{(1)})$$

(Note: If they are not iid, we lack homogeneity, independence, or both, and the standard LLN does not apply).

b) Convergence of the Instrument-Error Covariance (Orthogonality)

We assume that:

$$\frac{Z_n' \epsilon_n}{n} \xrightarrow{P} 0_{q\times 1}$$

This holds, for example, in the case where $(Z_{(i)}, \epsilon_{(i)})$ are iid, because Kolmogorov's LLN ensures that the sample average $\frac{1}{n}\sum_{i=1}^n Z_{(i)}' \epsilon_{(i)}$ converges in probability to zero.

4. Limit Function and Consistency of the IVE

Given conditions (a) and (b), and because the matrix transpose is an appropriately continuous operation, we have that $\frac{1}{n} X_n' Z_n \xrightarrow{P} Q_{Z'X}'$ and $\frac{1}{n}\epsilon_n' Z_n \xrightarrow{P} 0_{1\times q}$.

Therefore, we can prove that $M_n(\beta)$ will converge locally uniformly to a limit function $M(\beta)$, where the sample averages are replaced by their respective limits:

$$\begin{aligned} M(\beta) &= 0_{1\times q} W 0_{q\times 1} + (\beta_0 - \beta)' Q_{Z'X}' W Q_{Z'X} (\beta_0 - \beta) + 2(\beta_0 - \beta)' Q_{Z'X}' W 0_{q\times 1} \\ &= (\beta_0 - \beta)' Q_{Z'X}' W Q_{Z'X} (\beta_0 - \beta) \end{aligned}$$

Because $rank(Q_{Z'X}) = rank(Q_{Z'X}') = p$ and the weighting matrix W is positive definite, the inner matrix $Q_{Z'X}' W Q_{Z'X}$ is also strictly positive definite. *(This can be shown by decomposing $W = LL'$ using the Cholesky decomposition, giving us $Q_{Z'X}' L L' Q_{Z'X}$).*

Consequently, $M(\beta)$ takes the exact form of a squared Mahalanobis distance between β and β_0 with respect to the positive definite matrix $Q_{Z'X}' W Q_{Z'X}$.

If, additionally, Θ is a convex set (e.g., $\Theta = \mathbb{R}^p$), then β_0 is a **unique, well-separated minimizing point** of $M(\beta)$.

Therefore, we conclude that the Instrumental Variables estimator is weakly consistent:

$$\beta_n \xrightarrow{P} \beta_0$$

Classroom Q&A Highlight:

> **Student Question:** *Can the interval (the parameter space Θ) change with n?*
> **Professor's Answer:** *Yes, it is possible for it to change, but then we would explicitly have to discuss and prove the convergence of the Parameter Space itself.*