Econometrics 2

Lecture 11: Asymptotic Identification

Linear Model Example & Mild Misspecification

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1. The Condition of Asymptotic Identification

This well-separated minimum requirement we discussed previously is also referred to as the **condition of asymptotic identification**. Let us see how these concepts are specified in our examples.

2. Example: The Linear Model

Consider the standard objective function for the linear model:

$$M_n(\beta) = \frac{1}{n}(Y_n - X_n\beta)'(Y_n - X_n\beta)$$

We substitute the true data-generating process $Y_n = X_n\beta_0 + \epsilon_n$:

$$\begin{aligned}
M_n(\beta) &= \frac{1}{n}(\epsilon_n + X_n\beta_0 - X_n\beta)'(\epsilon_n + X_n\beta_0 - X_n\beta) \\
&= \frac{1}{n}\Big(\epsilon_n + X_n(\beta_0 - \beta)\Big)'\Big(\epsilon_n + X_n(\beta_0 - \beta)\Big) \\
&= \frac{1}{n}\Big(\epsilon_n'\epsilon_n + \epsilon_n'X_n(\beta_0 - \beta) + (\beta_0 - \beta)'X_n'\epsilon_n + (\beta_0 - \beta)'X_n'X_n(\beta_0 - \beta)\Big) \\
&= \frac{1}{n}\epsilon_n'\epsilon_n + \frac{2}{n}(\beta_0 - \beta)'X_n'\epsilon_n + \frac{1}{n}(\beta_0 - \beta)'X_n'X_n(\beta_0 - \beta)
\end{aligned}$$

Because only the terms containing β affect the optimization, we can drop the $\frac{1}{n}\epsilon_n'\epsilon_n$ term and keep the rest to define our simplified objective function for optimization:

$$\tilde{M}_n(\beta) = \frac{2}{n}(\beta_0 - \beta)'X_n'\epsilon_n + \frac{1}{n}(\beta_0 - \beta)'X_n'X_n(\beta_0 - \beta)$$

We will study the asymptotic behavior of the terms $\frac{X_n'X_n}{n}$ and $\frac{X_n'\epsilon_n}{n}$. It is proven that if these have limits in probability, then $\tilde{M}_n$ will converge locally uniformly in probability to an appropriate limit function $M(\beta)$ which will be determined by them.

3. Asymptotic Behavior of the Components

a) Convergence of the Gram Matrix
We assume that:

$$\frac{X_n' X_n}{n} \xrightarrow{P} Q_{X'X}$$

with $rank(Q_{X'X}) = p$ (meaning it does not depend on anything stochastic).

One way to ensure this is the following:

$$\frac{X_n' X_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_{(i)}' X_{(i)}$$

If the rows $X_{(i)}$ are iid and the expectation $\mathbb{E}(X_{(i)}' X_{(i)})$ exists, then Kolmogorov's Law of Large Numbers (LLN) applies. This implies that:

$$\frac{1}{n} \sum_{i=1}^{n} X_{(i)}' X_{(i)} \xrightarrow{P} \mathbb{E}(X_{(1)}' X_{(1)}) = Q_{X'X}$$

Additionally, $\mathbb{E}(X_{(1)}' X_{(1)})$ will have rank p if no regressor can be expressed as a linear combination of the remaining regressors.

b) Convergence of the Cross Product
We assume that:

$$\frac{1}{n} X_n' \epsilon_n \xrightarrow{P} 0_{p \times 1}$$

This will hold since $\frac{1}{n} X_n' \epsilon_n = \frac{1}{n} \sum_{i=1}^{n} X_{(i)}' \epsilon_{(i)}$. If the pairs $(X_{(i)}, \epsilon_{(i)})$ are iid and $\mathbb{E}(X_{(1)}' \epsilon_{(1)})$ exists, then the LLN applies again. Thus:

$$\frac{1}{n} \sum_{i=1}^{n} X_{(i)}' \epsilon_{(i)} \xrightarrow{P} \mathbb{E}(X_{(1)}' \epsilon_{(1)}) = 0_{p \times 1}$$

(We know this expectation is zero because the exogeneity assumption $\mathbb{E}(\epsilon_{(1)} | X_{(1)}) = 0$ *implies* $\mathbb{E}(X_{(1)}' \epsilon_{(1)}) = 0_{p \times 1}$*).*

4. The Limit Function and Unique Solutions

Given conditions (a) and (b), it is proven that our objective function converges:

$$\begin{aligned}
\tilde{M}_n(\beta) &= (\beta_0 - \beta)' \frac{X_n' X_n}{n} (\beta_0 - \beta) + 2(\beta_0 - \beta)' \frac{X_n' \epsilon_n}{n} \\
&\xrightarrow{P} (\beta_0 - \beta)' Q_{X'X} (\beta_0 - \beta) + 2(\beta_0 - \beta)' 0_{p \times 1} \\
&= (\beta_0 - \beta)' Q_{X'X} (\beta_0 - \beta) =: M(\beta)
\end{aligned}$$

Since $M(\beta)$ is a quadratic form involving $Q_{X'X}$, and $Q_{X'X}$ is invertible (full rank p), the resulting matrix is positive definite.

Taking the second derivative:

$$\frac{\partial^2 M(\beta)}{\partial\beta\partial\beta'} \propto Q_{X'X}$$

Therefore, the function $M(\beta)$ is **strictly convex**.

We can see that $\min_\beta M(\beta) = 0$, which occurs strictly when $(\beta_0 - \beta) = 0$. This implies $\beta = \beta_0$ is the **unique solution**. If the parameter space Θ is a convex set, then β_0 is a well-separated minimum.

5. "Mild" Misspecification: $\beta_0 \notin \Theta$

What happens if Θ is a convex subset of $\mathbb{R}^p$ but the true parameter β_0 does not lie within Θ? (This is known as "mild" misspecification).

Because we did not use the assumption that $\beta_0 \in \Theta$ during the derivation of $\tilde{M}_n(\beta)$, the convergence to the limit function $M(\beta)$ will still hold perfectly.

If it additionally holds that Θ is a convex and closed subset of $\mathbb{R}^p$, then $M(\beta)$ will still have a unique, well-separated minimizing point within Θ. Let us call this point $\beta^* \in \Theta$, where $\beta^* \neq \beta_0$.

It is proven that in this case, the estimator β_n will converge in probability to β^*. Consequently, the estimator **will not be consistent** for the true parameter β_0.

However, β^* maintains a specific geometric relationship with β_0: it will be the exact point within Θ that is at the **minimum distance** from β_0 based on the Mahalanobis distance metric defined by $Q_{X'X}$.