Econometrics 2

Lecture 3: Intuitive Analysis

Identification in Linear Models

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1. The Core Intuition

We want to uncover the true relationship, b_0, that generated our data. If we guess a parameter b, how wrong is our guess on average? The mathematical proof below demonstrates that our "average squared mistake" forms a perfect, multi-dimensional bowl.

Because of our statistical assumptions, the absolute bottom of this bowl occurs exactly when our guess matches reality ($b = b_0$). Keeping this "bowl" shape in mind helps anchor the algebra: every step is simply moving toward proving that the loss function is a parabola pointing upwards.

2. Model Setup

Our data generating process is defined as:

$$Y_n = X_n b_0 + \epsilon_n$$

The critical assumptions are:

- **Strict Exogeneity:** $\mathbb{E}(\epsilon_n|X_n) = 0_{n\times 1}$ (The errors are pure noise).
- **Spherical Errors:** $Var(\epsilon_n|X_n) = I_{n\times n}$ (The noise is uniform and uncorrelated).
- **Full Rank (Identification):** $rank(X_n) = p$ (We have enough unique information to pinpoint the parameters).

3. The Objective Function (The "Mistake" Function)

We define our expected loss function as:

$$U_n^*(b) := \mathbb{E}\left[\frac{1}{n}(Y_n - X_n b)'(Y_n - X_n b)\middle| X_n\right]$$

First, substitute the true model into our guess to see the error gap:

$$Y_n - X_n b = X_n b_0 + \epsilon_n - X_n b = X_n(b_0 - b) + \epsilon_n$$

Next, expand the quadratic form algebraically:

$$(X_n(b_0 - b) + \epsilon_n)'(X_n(b_0 - b) + \epsilon_n)$$

$$= (b_0 - b)'X_n'X_n(b_0 - b) + 2(b_0 - b)'X_n'\epsilon_n + \epsilon_n'\epsilon_n$$

4. Applying the Expectation (Filtering the Noise)

Now, we apply the conditional expectation operator $\frac{1}{n}\mathbb{E}(\cdot|X_n)$ to each of the three terms. This is where we mathematically "average out" the noise to reveal the underlying structure.

Term 1: The Deterministic Signal

Since b, b_0, and X_n are treated as constants under the condition, they pass through unchanged:

$$\frac{1}{n}\mathbb{E}\left[(b_0 - b)'X_n'X_n(b_0 - b)|X_n\right] = (b_0 - b)'\left(\frac{X_n'X_n}{n}\right)(b_0 - b)$$

Term 2: The Cross-Product (Zeroing Out)

Because the errors are pure noise ($\mathbb{E}(\epsilon_n|X_n) = 0$), this term vanishes. Recognizing that the expected value of the noise is zero immediately simplifies the expression:

$$\frac{1}{n}\mathbb{E}\left[2(b_0 - b)'X_n'\epsilon_n|X_n\right] = \frac{2}{n}(b_0 - b)'X_n'\mathbb{E}(\epsilon_n|X_n) = 0$$

Term 3: The Trace Trick (The Error Variance)

For a scalar value like $\epsilon_n'\epsilon_n$, the value equals its trace. Because the trace is a linear operator, we swap the trace and expectation to isolate the variance matrix:

$$\frac{1}{n}\mathbb{E}[\epsilon_n'\epsilon_n|X_n] = \frac{1}{n}\mathbb{E}[tr(\epsilon_n\epsilon_n')|X_n] = \frac{1}{n}tr(\mathbb{E}[\epsilon_n\epsilon_n'|X_n])$$

Since $Var(\epsilon_n|X_n) = I_{n\times n}$, the trace of an $n \times n$ identity matrix is simply n:

$$\frac{1}{n}tr(I_{n\times n}) = \frac{1}{n}(n) = 1$$

5. The Unique Minimum

Reassembling the filtered function gives us:

$$U_n^*(b) = (b_0 - b)' \left(\frac{X_n' X_n}{n} \right) (b_0 - b) + 1$$

Because X_n has full rank, the matrix $A = \frac{X_n' X_n}{n}$ is **positive definite**. This mathematically guarantees our "bowl" shape.

For any positive definite matrix A, the quadratic form $y'Ay \geq 0$, and it strictly equals 0 if and only if $y = 0_{p \times 1}$. Let $y = (b_0 - b)$. The minimum possible value of this entire function is uniquely achieved when:

$$b_0 - b = 0 \implies b = b_0$$

Therefore, minimizing this objective function cleanly and uniquely identifies the true parameter b_0:

$$\arg\min_{b \in \Theta} U_n^*(b) = b_0$$