4th Lecture: 3/11/2022

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• Statistical Testing of Linear Hypothesis:

Example:

Let $y = X\beta + \varepsilon$, where X has four columns

that is $k = 4$ and $\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$

We are interested in testing

$H_0 : \beta_2 = \beta_3 ,\ \beta_4 = 0$ vs $H_1 : \beta_2 \neq \beta_3$ and/or $\beta_4 \neq 0$.

In matrix form we have

$H_0 : R\beta = r$ vs $H_1 : R\beta \neq r$

where ~~R0 is the following 2x4~~ :

number of covariates + intercept

$R = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

* $R\beta \neq r$ means that at least one of the 2 restrictions is not true: either $\beta_2 \neq \beta_3$ and/or $\beta_4 \neq 0$

Formally,

$H_0 : R\beta = r \quad , \quad H_1 : R\beta \neq r$

where R is a $J \times k$ matrix with full row-rank, i.e, $\text{rank}(R) = J$. (*)

where J is the number of (linear) restrictions to be tested simultaneously.

(*) Why R to be of full row-rank?

Consider the same regression problem as before but with

and $H_0 : \beta_2 = \beta_3,\ \beta_4 = 0$ and $\beta_2 - \beta_3 = \beta_4$

the $J = \begin{array}{c} R1 \\ R2 \\ R3 \end{array} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$

but if we do $R1 - R2$ then we have $R3$, i.e. we have two independent rows $\Rightarrow$

$\Rightarrow \text{rank}(R) = 2$ or the third restriction is implied by the other two.

→ Statistic for $H_0: R\beta = r$ vs $H_1: R\beta \neq j$

We define the F-statistic as

$$F = \frac{(R\hat{\beta} - r)'\left(R(X'X)^{-1}R'\right)^{-1}(R\hat{\beta} - r)}{J\hat{\sigma}^2} \quad \text{wher } \hat{\sigma}^2 = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n-k}$$

then $F \sim F_{(J, n-k)}$

Proof: We have that

$$\hat{\beta} \mid X \sim N(\beta, \sigma^2 (X'X)^{-1})$$

$$\Rightarrow R\hat{\beta} \mid X \sim N(R\beta, \sigma^2 R(X'X)^{-1}R')$$

$$\overset{R\beta = r}{\Rightarrow} R\hat{\beta} \mid X \sim N(r, \sigma^2 R(X'X)^{-1}R')$$

So the ~~denominator~~ numerator in F is ~~the~~ a quadratic form of $N_J(0, I_J)$ random variable

Therefore $(\underset{J\times k}{R}\,\underset{k\times 1}{\hat{\beta}} - \underset{J\times 1}{r})'\left(R(X'X)^{-1}R'\right)^{-1}(R\hat{\beta} - r) \sim$

$$\sim \chi^2_J$$

However we have also proven that

$$\frac{\overset{\hat{\varepsilon}'\hat{\varepsilon}}{(n-k)\cdot\hat{\sigma}^2}}{\sigma^2} \sim \chi^2_{(n-k)}$$

Hint:

From staatistics we know that the ratio of two independent random variables χ_J^2 and χ_k^2 divided by their degrees of freedom, i.e., the random variable $F = \frac{Z/J}{W/k}$ where $Z \sim \chi_J^2$ and $W \sim \chi_k^2$ follows $F_{J,k}$

Therefore, we ~~write~~ multiply and divide F in previous page by $\frac{n-k}{\sigma^2}$ and

we write

$$F = \frac{(R\hat{\beta}-r)'\left(\frac{R(X'X)R'}{\sigma^2}\right)^{-1}(R\hat{\beta}-r)}{(n-k)\frac{\hat{\sigma}^2}{\sigma^2/n-k}} \sim$$

$$\sim F_{J,n-k}$$

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- Deciding for rejection of H_0 Based on the value of F

$$\hookrightarrow F = \frac{(R\hat{\beta} - r)' \left(R(X'X)^{-1}R'\right)^{-1} (R\hat{\beta} - r)}{J\hat{\sigma}^2} > 0$$

as ratio of quadratic forms

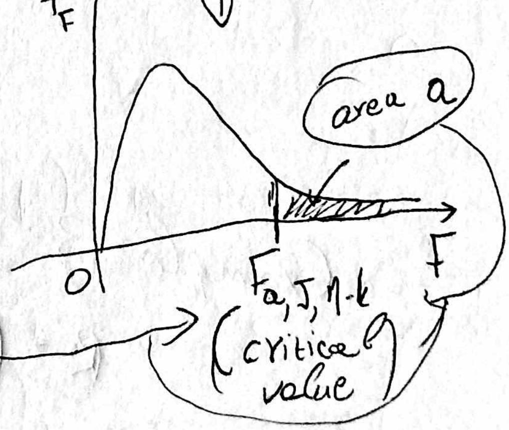
i) If $R\hat{\beta} - r \approx 0$ then we cannot reject H_0 and F small (or even close to zero)

ii) If $R\hat{\beta} - r$ large then we reject H_0 and F large

i) + ii) $\Rightarrow$ Reject H_0 if F takes values in the right tail of the $F_{J,n-k}$ distribution $\Rightarrow$

$\Rightarrow$ If $a = P(\text{wrong } H_0 \text{ rejection}) =$

$$= P\left(F_{J,n-k} > F_{a,J,n-k}\right)$$



- p-value for F-test

  ↳ Probability of extreme F under H_0 ⇒

⇒ If $P(F < F_{J,n-k})$ is small i.e. it is smaller than the prob. of wrong rejection ($=a$) then we should reject H_0.

Rule If p-value $< a$ then reject H_0

Remark: F-test can also be used for single linear restrictions

- Example 1: $H_0: \beta_j = 0$

  Then $\underset{1\times k}{R} = (0 \ldots \underset{\substack{\uparrow \\ j\text{-th} \\ \text{element}}}{1} 0 \ldots 0)$ and $r = 0$

- Example 2: $H_0: \beta_j = \beta_m \Rightarrow \beta_j - \beta_m = 0$

  ⇓

  $R = (0 \ldots \underset{j\text{th}}{1} 0 \ldots 0 \underset{m\text{th}}{-1} 0 \ldots 0)$

  and $r = 0$

- Example 3: $H_0: \beta_1 + \beta_3 = c$

  $R = (1\ 0\ 1\ 0 \ldots 0)$, $r = c$

→ Remarks on F-test.

- The F-test as studied by now is based on the so-called Wald's approach (we tested if $R\hat{\beta}$ is close to r).

- F test can be als derived as a likelihood ratio test; compare the value of the objective function under both H_0 (restricted estimation) an H_1 (unrestricted estimator)

→ $\min_{b} SSE(b)$ subject to $Rb = r$.

→ solution is called the restricted OLS estimator denoted as $\hat{\beta}_R$.

→ We can also define the corresponding residuals

$$\hat{\varepsilon}_R = y - X\hat{\beta}_R$$

and the restricte $SSE_R = \hat{\varepsilon}_R' \hat{\varepsilon}_R$

Then the F-statistic can take the form

$$F = \frac{(SSE_R - SSE)/J}{SSE/n-k}$$

comment on this

loss in explanatory power due to H_0 restrictions

- F-test for the overall significance of the Regression (BiCl lect 5, sl. 21) 8

↳ then may go to Asymp. Properties

- We need to test

$$H_0: \beta_1 = \beta_2 = .. \beta_{K-1} = 0$$

Then check that

H/W

$$F = \frac{R^2/K-1}{(1-R^2)/n-k}$$, R^2 is the R squared of the unrestricted regression