Differential Equations

Mathematics for Economists, Fall 2025-26

Homework Exercises Set 4

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1 Find the solution of the following differential equations, and determine the arbitrary constants:

a) $2y''(t) - 3y'(t) + y = (t^2+1)e^t, \; y(0) = 5, \; y'(0) = 14$

The solution of the homogeneous equations is:

$$y^h(t) = A_1 e^t + A_2 e^{0.5t}$$

Next, apply the Lagrange's method of variation of parameters. Let the following function with time varying coefficients $A_1(t)$ and $A_2(t)$ be a solution of the non-homogeneous equation:

$$y(t) = A_1(t)e^t + A_2(t)e^{0.5t}$$

$$\textit{Condition } 1: A_1'(t)e^t + A_2'(t)e^{0.5t} = 0$$

$$\textit{Condition } 2: A_1'(t)(e^t)' + A_2'(t)(e^{0.5t})' = \frac{(t^2+1)e^t}{2}$$

$$A_1'(t) = t^2 + 1$$

$$A_2'(t) = -(t^2+1)e^{0.5t}$$

$$A_1(t) = \frac{1}{3}t^3 + t$$

$$A_2(t) = -e^{0.5t}(2t^2 - 8t + 18)$$

Hence, the particular solution is:

$$\overline{y}(t) = (\frac{1}{3}t^3 + t)e^t + (-e^{0.5t}(2t^2 - 8t + 18))e^{0.5t}$$

Finding a different particular solution (by a different method) merely results in different constants A_1 and A_2 in the form of the general solution. We still get the same general solution.

The general solution is:

$$y(t) = 3e^t(t^3 - 6t^2 + 27t + 15)$$

b) $y''(t) + 2y'(t) + y = t^2, \; y(0) = 0, \; y'(0) = 1$

The solution of the homogeneous equations is:

$$y^h(t) = A_1 e^{-t} + A_2 t e^{-t}$$

The particular solution is:

$$\overline{y}(t) = t^2 - 4t + 6$$

The general solution is:

$$y(t) = -6e^{-t} - te^{-t} + t^2 - 4t + 6$$

2 Solve the following linear autonomous differential equations systems and construct their phase diagrams:

a) $\dot{x} = x + y,\ \dot{y} = 4x + y$

$$\lambda_u = 3 \qquad \lambda_s = -1$$

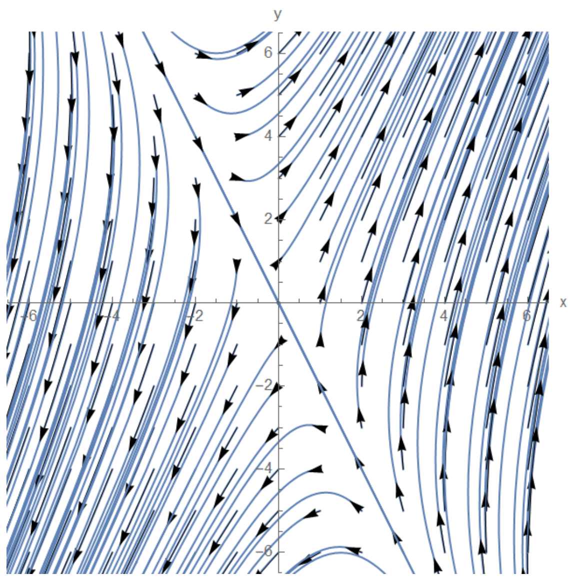
$$v_u = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \qquad v_s = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$$

$$s.s. \qquad (x^*, y^*) = (0, 0)$$

manifolds:

$$\frac{y - y^*}{x - x^*} = \frac{v_{u2}}{v_{u1}} = 2 \Rightarrow y = 2x \qquad unstable\ manifold$$

$$\frac{y - y^*}{x - x^*} = \frac{v_{s2}}{v_{s1}} = -2 \Rightarrow y = -2x \qquad stable\ manifold$$



b) $\dot{x} = 3x - 2y,\ \dot{y} = 2x - 2y$

$$\lambda_u = 2 \qquad \lambda_s = -1$$

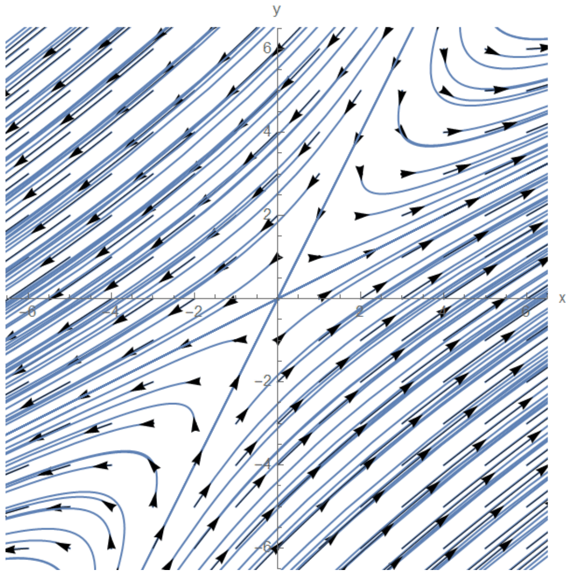
$$v_u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad v_s = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$s.s. \qquad (x^*, y^*) = (0, 0)$$

manifolds:

$$\frac{y - y^*}{x - x^*} = \frac{v_{u2}}{v_{u1}} = 0.5 \Rightarrow y = 0.5x \qquad unstable\ manifold$$

$$\frac{y - y^*}{x - x^*} = \frac{v_{s2}}{v_{s1}} = 2 \Rightarrow y = 2x \qquad stable\ manifold$$



3 Solve the equilibrium for the following control problem. Linearise the system about the equilibrium and establish its stability properties:

$$\max_{\{u\}} J = \int_0^\infty (20 ln x - 0.1u^2)dt$$

$$s.t. \quad \dot{x} = u - 0.1x$$

The Hamiltonian for this problem is:

$$H = 20 ln x - 0.1u^2 + \lambda(u - 0.1x)$$

with first-order conditions:

$$\frac{\partial H}{\partial u} = -0.2u + \lambda = 0$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -(\frac{20}{x} - 0.1\lambda)$$
$$\dot{x} = u - 0.1x$$

which can be reduced to two differential equations in terms of x and λ:

$$\dot{x} = -0.1x + 5\lambda$$

$$\dot{\lambda} = -\frac{20}{x} + 0.1\lambda$$

The fixed point of this system is found by setting $\dot{x} =$ and $\dot{\lambda} = 0$, giving $x^* = 100$ and $\lambda^* = 2$. Furthermore, the two isoclines are found to be:

$$\lambda = 0.02x \qquad (\dot{x} = 0)$$

$$\lambda = \frac{200}{x} \qquad (\dot{\lambda} = 0)$$

Considering a linearisation about the fixed point $(x^*, \lambda^*) = (100, 2)$.
The Jacobian matrix of the system:

$$J = \begin{bmatrix} -0.1 & 5 \\ \frac{20}{x^2} & 0.1 \end{bmatrix}$$

evaluated at the equilibrium point $(x^*, \lambda^*) = (100, 2)$.

$$J_{(100,2)} = \begin{bmatrix} -0.1 & 5 \\ 0.002 & 0.1 \end{bmatrix}$$

This gives the linear equations:

$$\dot{x} = -0.1(x - x^*) + 5(\lambda - \lambda^*)$$

$$\dot{\lambda} = 0,002(x - x^*) + 0.1(\lambda - \lambda^*)$$

The eigenvalues of matrix J is:

$$\lambda_s = -0.14142 \qquad \lambda_u = 0.14142$$

Which indicate a saddle point solution since the Hartman–Grobman theorem states that the behaviour of a dynamical system in a domain near a hyperbolic equilibrium point is qualitatively the same as the behaviour of its linearisation near this equilibrium point.