Systems of Difference Equations

Mathematics for Economists, Fall 2025-26

Homework Exercises Set 3

Teaching Assistant: Maria Gioka

Email: gkiokam@aueb.gr

1 Express the following n-order linear difference equations as a first-order linear $(n\times n)$ difference equations systems:

a) $y_{t+3}+5y_{t+2}-4y_{t+1}+y_t=t$

$$\begin{bmatrix} z_{t+1} \\ x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} -5 & 4 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \\ y_t \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

b) $y_t+y_{t-2}+0.25y_{t-4}=0$

$$\begin{bmatrix} y_t \\ x_t \\ z_t \\ w_t \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -0.25 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \\ z_{t-1} \\ w_{t-1} \end{bmatrix}$$

c) $y_{t+4}+5y_{t+2}+4y_t=0$

$$\begin{bmatrix} w_{t+1} \\ z_{t+1} \\ x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & -5 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_t \\ z_t \\ x_t \\ y_t \end{bmatrix}$$

2 Solve the following first order difference systems, using the direct method:

a)

$x_{t+1}=-x_t+y_t-8$

$y_{t+1}=-0.3x_t+0.9y_t+4$

Eigenvalues $\lambda_1=0.73$, $\lambda_2=-0.83$. The system is stable. $v_1=\begin{bmatrix} 0.58 \\ 1 \end{bmatrix}$, $v_2=\begin{bmatrix} 5.75 \\ 1 \end{bmatrix}$, Particular solution $\begin{bmatrix} 6.4 \\ 20.8 \end{bmatrix}$

$$x_t=c_1 0.58\lambda_1^t+c_2 5.75\lambda_2^t+6.4$$

$$y_t=c_1\lambda_1^t+c_2\lambda_2^t+20.8$$

b)

$x_{t+1}=x_t-y_t$

$y_{t+1}=x_t+3y_t$

Characteristic equation: $\lambda^2-4\lambda+4=0$, $\lambda_1=\lambda_2=\lambda=2>1$. The system is unstable.

$$\begin{bmatrix} 1-2 & -1 \\ 1 & 3-2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -v_{11}-v_{12}=0 \\ v_{11}+v_{12}=0 \end{matrix}$$

So, we have that $v_{11}=-v_{12}$. Therefore

$$v_1=\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} -v_{12} \\ v_{12} \end{bmatrix} = v_{12}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So, for $v_{12} = 1$ the independent eigenvector corresponding to $\lambda_1 = 2$ is

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

while the generalized eigenvector v_2 is

$$(A - \lambda)v_2 = v_1$$

where v_1 is the independent eigenvector. So we have that:

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{matrix} -v_{21} - v_{22} = -1 \\ v_{21} + v_{22} = 1 \end{matrix} \quad \rightarrow v_{21} = 1 - v_{22}$$

Therefore, for $v_{22} = 1$ the generalized eigenvector is:

$$v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 - v_{22} \\ v_{22} \end{bmatrix} \overset{v_{22}=1}{=} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The general solution of the system is:

$$x_t = c_1 v_{11} \lambda^t + c_2 t v_{11} \lambda^{t-1} + c_2 v_{21} \lambda^t = -c_1 2^t - c_2 t 2^{t-1}$$

$$y_t = c_1 v_{12} \lambda^t + c_2 t v_{12} \lambda^{t-1} + c_2 v_{22} \lambda^t = c_1 2^t + c_2 t 2^{t-1} + c_2 2^t$$

where c_1 and c_2 are arbitrary constants.