MSc MATHS ECON - Tutorial 3

Economic applications & FOD Systems

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Economic application

Consider Samuelson's (1939) multiplier-accelerator model of income determination:

$$C_t = bY_{t-1} \qquad 0 < b < 1 \tag{1}$$

$$I_t = I'_t + I''_t \tag{2}$$

$$I'_t = k(C_t - C_{t-1}) \qquad k > 0 \tag{3}$$

$$I''_t = G \qquad G = constant \tag{4}$$

$$Y_t = C_t + I_t \tag{5}$$

The consumption equation (1) is a linear function of lagged income where b stands for the marginal propensity to consume (MPC). Income has two components, namely induced investment, I'_t, and autonomous investment, I''_t, equals to public spending, G,assumed constant at all periods (strictly exogenous). Induced income evolves according to the acceleration equation (3) where k is the acceleration coefficient.

Solving the model (substituting and separating endogenous form exogenous variables) yields:

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$$Y_t = C_t + I_t$$

$$Y_t = bY_{t-1} + I_t' + I_t''$$

$$Y_t = bY_{t-1} + k(C_t - C_{t-1}) + G$$

$$Y_t = bY_{t-1} + k(bY_{t-1} - bY_{t-2}) + G$$
$$Y_t - b(1+k)Y_{t-1} + kbY_{t-2} = G$$

This is linear second order difference equation in Y_t.

Since its RHS is constant an appropriate guess function would be $Y_t = \mu$. Substituting into the second order difference equation we have

$$\mu - b(1+k)\mu + kb\mu = G \Leftrightarrow \mu = G\frac{1}{1-b}$$

which is the particular solution i.e. **the long run equilibrium**. The solution to the homogeneous equation is a function of the roots of the following characteristic polynomial:

$$\lambda^2 - b(1+k)\lambda + bk = 0$$

$$\lambda^2 + a_1\lambda + a_2 = 0$$

$$a_1 \equiv -b(1+k) \qquad a_2 \equiv bk$$

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So,

$$a_1 \equiv -b(1+k) \qquad a_2 \equiv bk$$

These coefficients satisfy the stability conditions stated in (5.19) on p.58 in Gandolfo:

$$1 + a_1 + a_2 > 0 \Rightarrow 1 - b(1+k) + bk > 0 \Rightarrow 1 - b > 0$$

$$1 - a_2 > 0 \Rightarrow 1 - bk > 0$$

$$1 - a_1 + a_2 > 0 \Rightarrow 1 + b(1+k) + bk > 0$$

The second condition requires that:

$$bk < 1 \quad or \quad b < \frac{1}{k}$$

Moreover according to Descartes' theorem (Gandolfo p.54), since the coefficients of the characteristic equation alternate in sign, no negative root may occur, hence improper oscillations are excluded.

The quality of the roots depends on the discriminant, Δ:

$$\Delta \lesseqqgtr 0$$

$$b^2(1+k)^2 - 4bk \lesseqqgtr 0 \qquad b > 0$$

$$b \lesseqqgtr \frac{4k}{(1+k)^2}$$

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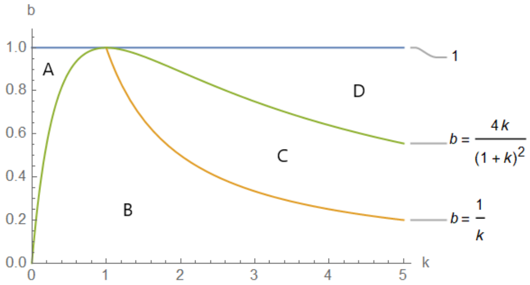

Figure: Samuelson's multiplier-accelerator diagram.

Consider the following rational expectations model:

$$C_t = -\beta p_t, \qquad \beta > 0 \tag{6}$$

$$Q_t = \gamma p_t^e + z_t, \qquad \gamma > 0 \tag{7}$$

$$I_t = a(p_{t+1}^e - p_t), \qquad a > 0 \tag{8}$$

$$C_t + I_t = Q_t + I_{t-1} \tag{9}$$

$$p_t^e = p_t, \qquad p_{t+1}^e = p_{t+1} \tag{10}$$

Equation (6) expresses current consumption (demand) as a function of current price, p_t. Equation (7) expresses current production (supply) as a function of the price expected to hold in the current period, p_t^e, while z_t is an exogenous (deterministic) supply shock. Equation (8) is the current inventory level held for speculative purposes. Equation (10) is the transition equation or adjustment equation. It assumes rational expectations i.e. mean perfect foresight. Substituting equations (6) - (8) into (9) using (10) yields the following sode:

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$$-\beta p_t + a(p^e_{t+1} - p_t) = \gamma p^e_t + z_t + a(p^e_t - p_{t-1})$$

$$-\beta p_t + a(p_{t+1} - p_t) = \gamma p_t + z_t + a(p_t - p_{t-1})$$

$$-\beta p_t + ap_{t+1} - ap_t = \gamma p_t + z_t + ap_t - ap_{t-1}$$

$$ap_{t+1} - 2ap_t - \beta p_t - \gamma p_t + ap_{t-1} = z_t$$

$$ap_{t+1} - (2a + \beta + \gamma)p_t + ap_{t-1} = z_t$$

$$p_{t+1} - (\frac{2a + \beta + \gamma}{a})p_t + p_{t-1} = \frac{z_t}{a}$$

Lagging once and setting $X_t \equiv \frac{z_{t-1}}{a} \equiv L^1(\frac{z_t}{a})$ yields:

$$p_t - (\frac{2a + \beta + \gamma}{a})p_{t-1} + p_{t-2} = X_t$$

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The characteristic polynomial of the homogeneous equation is

$$\lambda^2 - (\frac{2a + \beta + \gamma}{a})\lambda + 1 = 0$$

$$\Delta = (\frac{2a + \beta + \gamma}{a})^2 - 4 = \frac{(2a + \beta + \gamma)^2 - 4a^2}{a^2} > 0$$

Given $\Delta > 0$, applying Descartes' theorem, both roots will be positive (the coefficients of the characteristic polynomial alternate in sign). Thus, the solution of the homogeneous equation equals

$$p_t^h = A_1\lambda_1^t + A_2\lambda_2^t$$

Since the product of the roots equals unity, $\lambda_1\lambda_2 = 1$, they are inverse related i.e. $\lambda_1 = \frac{1}{\lambda_2}$. Let $\lambda_1 < 1$ implying that $|\lambda_1| < 1$ so λ_1 is the stable root. Them, $\lambda_2 > 1$ implying that $|\lambda_2| > 1$ so λ_2 is the unstable root. Therefore, the solution is unstable.

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Next, use lag operators in order to find the solution of the non-homogeneous equation

$$p_t - (\frac{2a + \beta + \gamma}{a})p_{t-1} + p_{t-2} = X_t$$

$$p_t - (\frac{2a + \beta + \gamma}{a})Lp_t + L^2 p_t = X_t$$

Thus the particular solution equals:

$$\bar{p}_t = [\frac{1}{1 - (\frac{2a+\beta+\gamma}{a})L + L^2}]X_t = (\frac{\theta_1}{1 - \lambda_1 L} + \frac{\theta_2}{1 - \lambda_2 L})X_t$$

where constants θ_1 and θ_2 are obtained by solving for the partial fractions of the rational polynomial above:

$$\theta_1 = \frac{\lambda_1}{\lambda_1 - \lambda_2}, \quad \theta_2 = \frac{-\lambda_2}{\lambda_1 - \lambda_2},$$

Since $|\lambda_1| < 1$ and $|\lambda_2| > 1$, the backward expansion should be used for λ_1 and the forward expansion for λ_2. In this way, the particular solution will have a form of two-sided distributed lag, i.e. of a weighted sum of past, present, and future values of X_t:

$$\bar{p}_t = (\frac{\theta_1}{1 - \lambda_1 L} + \frac{\theta_2}{1 - \lambda_2 L})X_t$$

$$\bar{p}_t = \theta_1 \sum_{i=0}^{\infty} \lambda_1^i X_{t-i} - \theta_2 \sum_{i=1}^{\infty} (\frac{1}{\lambda_2})^i X_{t+i}$$

Moreover, since $\frac{1}{\lambda_2} = \lambda_1$:

$$\bar{p}_t = \theta_1 \sum_{i=0}^{\infty} \lambda_1^i X_{t-i} - \theta_2 \sum_{i=1}^{\infty} \lambda_1^i X_{t+i}$$

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Thus the general solution equals:

$$p_t = A_1\lambda_1^t + A_2\lambda_2^t + \bar{p}_t$$

$$p_t = A_1\lambda_1^t + A_2\lambda_1^{-t} + \bar{p}_t$$

where $|\lambda_1| < 1$ and $|\lambda_2| > 1$

In order to determine the constants A_1 and A_2 we will impose boundary conditions on the entire path of p_t for all bounded sequences of the exogenous shock, $\{X_t\}$. Observe that:

$$\lim_{t\to\infty} \lambda_1^t = 0 \wedge \lim_{t\to\infty} \lambda_2^t = \lim_{t\to\infty} \lambda_1^{-t} = +\infty \Rightarrow A_2 = 0$$

$$\lim_{t\to-\infty} \lambda_1^t = +\infty \wedge \lim_{t\to-\infty} \lambda_2^t = \lim_{t\to-\infty} \lambda_1^{-t} = 0 \Rightarrow A_1 = 0$$

Thus, the entire path will be bounded as long as $A_1 = A_2 = 0$:

$$\lim_{t\to\pm\infty} |p_t| < +\infty \Leftrightarrow A_1 = A_2 = 0$$

Higher-order Difference equation (lags)

Express the following linear difference equation as a linear first-order difference system:

$$y_t - 4y_{t-1} + 4.8y_{t-2} - 1.6y_{t-3} = 100$$

Since the polynomial order is $p = 3$, define $p - 1 = 2$ new variables, say x_t and z_t :

$$x_t \equiv y_{t-1}$$

$$z_t \equiv y_{t-2} \equiv x_{t-1}$$

$$z_{t-1} = y_{t-3}$$

Substituting in the original and solving for y_t yields:

$$y_t = 4x_t - 4.8z_t + 1.6z_{t-1} + 100$$

$$y_t = 4y_{t-1} - 4.8x_{t-1} + 1.6z_{t-1} + 100$$

Higher-order Difference equation (lags)

$$\begin{bmatrix} y_t \\ x_t \\ z_t \end{bmatrix} = \begin{bmatrix} 4 & -4.8 & 1.6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

Stacking yields a linear first-order $p \times p$ difference system:

$$\underset{p\times 1}{Y_t} = \underset{p\times p}{A}\,\underset{p\times 1}{Y_{t-1}} + \underset{p\times 1}{C}$$

Higher-order Difference equation (leads)

Express the following linear difference equation as a linear first-order difference system:

$$y_{t+3} - y_{t+2} - 2y_{t+1} + 2y_t = 0$$

Since the polynomial order is $p = 3$, define $p - 1 = 2$ new variables, say x_t and z_t :

$$x_t \equiv y_{t+1}$$

$$z_t \equiv y_{t+2} \equiv x_{t+1}$$

$$z_{t+1} = y_{t+3}$$

Substituting in the original yields:

$$z_{t+1} = z_t + 2x_t - 2y_t$$

Higher-order Difference equation (leads)

$$\begin{bmatrix} z_{t+1} \\ x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \\ y_t \end{bmatrix}$$

Stacking yields a linear first-order $p \times p$ difference system:

$$\underset{p\times 1}{Y_{t+1}} = \underset{p\times p}{A}\,\underset{p\times 1}{Y_t} + \underset{p\times 1}{C}$$

Solve the following FODS:

$$\left\{ \begin{array}{c} x_{t+1} = x_t + 2y_t + 2 \\ y_{t+1} = 4x_t + 3y_t + 1 \end{array} \right\}$$

This is a linear 2×2 system in x_t and y_t, already in normal form. Defining:

$$X_{t+1} := \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix}, \quad X_t := \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \quad A := \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$G := \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad X_0 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We may equivalently write the system in vector form:

$$X_{t+1} = AX_t + G$$

Apply the direct method and find the characteristic polynomial of square matrix A:

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) - 8 = 0 \Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

Since $\Delta = 36 > 0$ the polynomial has a pair of real distinct roots (eigenvalues), $\lambda_1 = 5$, $\lambda_2 = -1$. Since the eigenvalues are distinct, the eigenvector $v_1 = (v_{11}, v_{12})$, $v_2 = (v_{21}, v_{22})$ will be linear independent:

$$\begin{bmatrix} 1-5 & 2 \\ 4 & 3-5 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} -4v_{11} + 2v_{12} = 0 \\ 4v_{11} - 2v_{12} = 0 \end{array}$$

So, we have that $v_{11} = 0.5v_{12}$.Therefore

$$v_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0.5v_{12} \\ v_{12} \end{bmatrix} = v_{12} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

So, an eigenvector corresponding to $\lambda_1 = 5$ is

$$v_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

while the eigenvector corresponding to $\lambda_2 = -1$ is

$$\begin{bmatrix} 1-(-1) & 2 \\ 4 & 3-(-1) \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 2v_{21} + 2v_{22} = 0 \\ 4v_{21} + 4v_{22} = 0 \end{array}$$

So, we have that $v_{21} = -v_{22}$.Therefore

$$v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} -v_{22} \\ v_{22} \end{bmatrix} = v_{22} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So, an eigenvector corresponding to $\lambda_2 = -1$ is

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The general solution of the homogeneous system is:

$$x_t = c_1 v_{11} \lambda_1^t + c_2 v_{21} \lambda_2^t = c_1 0.5 \lambda_1^t + c_2 (-1) \lambda_2^t$$

$$y_t = c_1 v_{12} \lambda_1^t + c_2 v_{22} \lambda_2^t = c_1 1 \lambda_1^t + c_2 1 \lambda_2^t$$

where c_1 and c_2 are arbitrary constants

Real distinct roots ($\Delta > 0$)

We can now turn to the problem of finding a particular solution of the non-homogeneous system. The method of undetermined coefficients can be applied here too. Since $G = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ let us try $\bar{X} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$, where $\bar{x}_1$ and $\bar{x}_2$ are undetermined constants.

$$\bar{X} = A\bar{X} + G$$

$$(I - A)\bar{X} = G$$

$$\bar{X} = (I - A)^{-1} G$$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 1-1 & -2 \\ -4 & 1-3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -4 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{-8} \begin{bmatrix} -2 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.25 & -0.25 \\ -0.5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ -1 \end{bmatrix}$$

Real distinct roots ($\Delta > 0$)

Use the the initial condition vector in order to determine the vector of arbitrary constants in the general solution:

$$t = 0 : \left\{ \begin{array}{c} x_0 = c_1 0.5\lambda_1^0 + c_2(-1)\lambda_2^0 + 0.25 \\ y_0 = c_1 1\lambda_1^0 + c_2 1\lambda_2^0 - 1 \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{c} 1 = 0.5c_1 - c_2 + 0.25 \\ 1 = c_1 + c_2 - 1 \end{array} \right\}$$

$$C = \left[\begin{array}{c} c_1 \\ c_2 \end{array} \right] = \left[\begin{array}{c} 0.5 \\ -0.5 \end{array} \right]$$

Real repeated root (Δ=0)

Solve the following FODS

$$\left\{ \begin{array}{c} y1_{t+1} = -3y1_t - 2y2_t - 2a^t \\ y2_{t+1} = 2y1_t + y2_t + a^t \end{array} \right\}$$

This is a linear 2×2 system in $y1_t$ and $y2_t$, in normal form, where

$$A := \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}, \quad G(t) := Ga^t = \begin{bmatrix} -2 \\ 1 \end{bmatrix} a^t, \quad Y_0 := \begin{bmatrix} y1_0 \\ y2_0 \end{bmatrix}$$

We may equivalently write the system in vector form:

$$Y_{t+1} = AY_t + G(t)$$

Real repeated root (Δ=0)

Apply the direct method and find the characteristic polynomial of square matrix A:

$$\begin{vmatrix} -3-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (-3-\lambda)(1-\lambda)+4 = 0 \Rightarrow \lambda^2+2\lambda+1 = 0$$

Since $\Delta = 0$ the polynomial has a real repeated root (eigenvalue), $\lambda = \lambda_1 = \lambda_2 = -1$. Because $\lambda \not< 1$ the system is unstable. The independent eigenvector v_1 is:

$$\begin{bmatrix} -3-(-1) & -2 \\ 2 & 1-(-1) \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -2v_{11} - 2v_{12} = 0 \\ 2v_{11} + 2v_{12} = 0 \end{matrix}$$

So, we have that $v_{11} = -v_{12}$.Therefore

$$v_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} -v_{12} \\ v_{12} \end{bmatrix} = v_{12} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So, the independent eigenvector corresponding to $\lambda_1 = -1$ is

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Real repeated root ($\Delta=0$)

while the generalized eigenvector v_2 is

$$(A - \lambda \mathrm{I})v_2 = v_1$$

where v_1 is the independent eigenvector. So we have that:

$$\begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{matrix} -2v_{21} - 2v_{22} = -1 \\ 2v_{21} + 2v_{22} = 1 \end{matrix} \rightarrow v_{21} = 0.5 - v_{22}$$

Therefore, the generalized eigenvector is:

$$v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0.5 - v_{22} \\ v_{22} \end{bmatrix} \overset{v_{22}=1}{=} \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$$

Real repeated root (Δ=0)

The general solution of the homogeneous system is:

$$y1_t = c_1 v_{11}\lambda^t + c_2 t v_{11}\lambda^{t-1} + c_2 v_{21}\lambda^t = [-c_1 + c_2(t-0.5)](-1)^t$$

$$y2_t = c_1 v_{12}\lambda^t + c_2 t v_{12}\lambda^{t-1} + c_2 v_{22}\lambda^t = [c_1 - c_2(t-1)](-1)^t$$

where c_1 and c_2 are arbitrary constants.
The non-homogeneous part, $G(t)$, is a known functional form. Try the exponential guess function:

$$\overline{Y}_t = Ka^t, \qquad \overline{Y}_{t+1} = Ka^{t+1}$$

$$\overline{Y}_{t+1} = A\overline{Y}_t + G(t)$$

$$Ka^{t+1} = AKa^t + Ga^t$$

$$Ka = AK + G$$

Real repeated root (Δ=0)

$$K = AKa^{-1} + Ga^{-1}$$

$$K - AKa^{-1} = Ga^{-1}$$

$$K = (I - Aa^{-1})^{-1} Ga^{-1}$$

$$K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -\frac{3}{a} & -\frac{2}{a} \\ \frac{2}{a} & \frac{1}{a} \end{bmatrix} \right)^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \frac{1}{a} = \begin{bmatrix} \frac{-2a}{(a+1)^2} \\ \frac{a-1}{(a+1)^2} \end{bmatrix}$$

Therefore:

$$GS = CF + PS$$

$$y1_t = [-c_1 + c_2(t - 0.5)](-1)^t - \frac{2a}{(a+1)^2} a^t$$

$$y2_t = [c_1 - c_2(t - 1)](-1)^t + \frac{a-1}{(a+1)^2} a^t$$