Lecture 13

endogenous variables (n)

$$x_t = \begin{bmatrix} z_t \\ p_t \end{bmatrix} \xrightarrow[n-m]{} \text{state (predetermined at } t\text{)}$$

$n \times 1$ $\quad \longrightarrow m$ control (jump / free to choose at time t)

exogenous variables (r)

u_t
$r \times 1$

Under perfect foresight

$$x_{t+1} = A \cdot x_t + B \cdot u_t$$

solving forwards:

SOLUTION

$$x_{t+k} = \underbrace{A^k \cdot x_t}_{\text{solution of homogeneous}} + \underbrace{\sum_{j=0}^{k-1} A^{k-1-j} B \cdot u_{t+j}}_{\text{particular solution}}$$

At time t → z_t is known
→ p_t is not known

$$\underset{n\times 1}{x_{t+1}} = \underset{n\times n}{A} \cdot \underset{n\times 1}{x_t} + \underset{n\times r}{B} \cdot \underset{r\times 1}{u_t}$$

n eigenvalues λ_i of A — ℓ unstable $|\lambda_i| > 1$, $i = 1..\ell$
— $n-\ell$ stable $|\lambda_i| < 1$, $i = \ell+1,..,n$

(1)

Assume $A = T^{-1} \cdot \Lambda \cdot T$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

Partition: $\Lambda = \begin{bmatrix} \lambda_1 & & \emptyset \\ & \lambda_2 & \\ \emptyset & & \ddots \; \lambda_n \end{bmatrix} = \underset{n\times n}{\begin{bmatrix} \Lambda_1 & \emptyset \\ \emptyset & \Lambda_2 \end{bmatrix}}$ $\quad \underset{(n-\ell)\times(n-\ell)}{\Lambda_1} \to$ stable eigenvalues, $\underset{\ell\times\ell}{\Lambda_2} \to$ unstable "

matrix of eigenvectors

$$T = \underset{n\times n}{\begin{bmatrix} \underset{(n-\ell)\times n}{T_1} \\ \underset{\ell\times n}{T_2} \end{bmatrix}}$$

$$T^{-1} = \underset{n\times n}{\begin{bmatrix} \underset{n\times(n-\ell)}{S_1} & \underset{n\times\ell}{S_2} \end{bmatrix}}$$

$$A^k = T^{-1} \Lambda^k T = T^{-1} \begin{bmatrix} \lambda_1^k & & \emptyset \\ & \lambda_2^k & \\ \emptyset & & \ddots \; \lambda_n^k \end{bmatrix} T = T^{-1} \begin{bmatrix} \Lambda_1^k & \emptyset \\ \emptyset & \Lambda_2^k \end{bmatrix} T$$

Solution: $x_{t+k} = A^k \cdot x_t + \sum_j A^{k-1-j} B \cdot u_{t+j} =$

$$= T^{-1} \cdot \begin{bmatrix} \Lambda_1^k & \emptyset \\ \emptyset & \Lambda_2^k \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} x_t + T^{-1} \left\{ \sum_j \begin{bmatrix} \Lambda_1^{k-1-j} & \emptyset \\ \emptyset & \Lambda_2^{k-1-j} \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \cdot B \cdot u_{t+j} \right\} =$$

$$= \left\{ \begin{aligned} T^{-1} \Lambda^k T &= [S_1 \; S_2] \cdot \begin{bmatrix} \Lambda_1^k & \emptyset \\ \emptyset & \Lambda_2^k \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \\ &= [S_1 \; S_2] \cdot \begin{bmatrix} \Lambda_1^k \cdot T_1 \\ \Lambda_2^k \, T_2 \end{bmatrix} = \\ &= S_1 \Lambda_1^k \cdot T_1 + S_2 \cdot \Lambda_2^k \cdot T_2 \end{aligned} \right\} =$$

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endogenous variables (n)

$$\underset{n\times 1}{x_t} = \begin{bmatrix} z_t \\ p_t \end{bmatrix} \begin{matrix} \xrightarrow[n-m]{} \text{state (predetermined at } t) \\ \rightarrow m \text{ control (jump / free to choose at time } t) \end{matrix}$$

exogenous variables (r)

$\underset{r\times 1}{u_t}$

Under perfect foresight

$$x_{t+1} = A \cdot x_t + B \cdot u_t$$

solving forwards:

$\boxed{\text{SOLUTION}}$

$$x_{t+k} = \underbrace{A^k \cdot x_t}_{\text{solution of homogeneous}} + \underbrace{\sum_{j=0}^{k-1} A^{k-1-j} B \cdot u_{t+j}}_{\text{particular solution}}$$

At time t $\nearrow$ z_t is known
$\searrow$ p_t is not known

$$\underset{n\times 1}{x_{t+1}} = \underset{n\times n}{A} \cdot \underset{n\times 1}{x_t} + \underset{n\times r}{B} \cdot \underset{r\times 1}{u_t}$$

λ_i n eigenvalues of A $\nearrow$ l unstable $|\lambda_i| > 1$, $i = 1 \dots l$
$\searrow$ $n-l$ stable $|\lambda_i| < 1$, $i = l+1, \dots, n$

Partition: $\Lambda = \begin{bmatrix} \Lambda_1 & \emptyset \\ \emptyset & \Lambda_2 \end{bmatrix}$

matrix of eigenvector

$$T = \begin{bmatrix} \underset{(n-l)\times n}{T_1} \\ \underset{l\times n}{T_2} \end{bmatrix}_{n\times n}$$

$$T^{-1} = \begin{bmatrix} \underset{n\times(n-l)}{S_1} & \underset{n\times l}{S_2} \end{bmatrix}_{n\times n}$$

$$A^k = T^{-1} \Lambda^k T = T^{-1} \begin{bmatrix} \lambda_1^k & & \emptyset \\ & \lambda_2^k & \\ \emptyset & & \ddots \lambda_n^k \end{bmatrix} T = T^{-1} \begin{bmatrix} \Lambda_1^k & \emptyset \\ \emptyset & \Lambda_2^k \end{bmatrix} T$$

Therefore

$$x_{t+k} = \left[S_1 \cdot \Lambda_1^k T_1 + S_2 \cdot \Lambda_2^k T_2 \right] \cdot x_t + \sum_j \left[S_1 \Lambda_1^{k-1-j} T_1 + S_2 \Lambda_2^{k-1-j} T_2 \right] B \cdot u_{t+j}$$

$$\Leftrightarrow x_{t+k} = \underbrace{S_1 \Lambda_1^k \cdot \left\{ T_1 \cdot x_t + \sum_j \Lambda_1^{-1-j} T_1 \cdot B \cdot u_{t+j} \right\}}_{\mathrm{I}} + \underbrace{S_2 \cdot \Lambda_2^k \cdot \left\{ T_2 \cdot x_t + \sum_j \Lambda_2^{-1-j} T_2 \cdot B \cdot u_{t+j} \right\}}_{\mathrm{II}}$$

$\lim_{k\to\infty}$

$\lim_{k\to\infty} \Lambda_1^k \to 0$ $\quad$ $\mathrm{I} \to 0$

$\lim_{k\to\infty} \Lambda_2^k \to \infty$, $\quad$ $\mathrm{II} \to \infty$

Unless

$$\left\{ T_2 \cdot x_t + \sum \Lambda_2^{-1-j} T_2 B \cdot u_{t+j} \right\} = 0$$

In this case $\mathrm{II} = 0$

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$$x_{t+1} = A \cdot x_t + B \cdot u_t$$

$n\times1 \quad n\times n \quad n\times1 \quad n\times r \quad r\times1$

λ_i n eigenvalues of A

- ℓ unstable $|\lambda_i| > 1$, $i = 1 \ldots \ell$
- $n-\ell$ stable $|\lambda_i| < 1$, $i = \ell+1, \ldots, n$

endogenous variables (n)

$$\underset{n\times1}{x_t} = \begin{bmatrix} z_t \\ p_t \end{bmatrix} \begin{matrix} \to \text{state (predetermined) at } t \quad (n-m) \\ \to m \text{ control (jump / free to choose at time } t) \end{matrix}$$

exogenous variables (r)

$\underset{r\times1}{u_t}$

$$T_2 = [T_{21} \; T_{22}]$$

$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

$$T = \begin{bmatrix} \underset{(n-\ell)\times\ell}{T_{11}} & \underset{(n-\ell)\times\ell}{T_{12}} \\ \underset{\ell\times(n-m)}{T_{21}} & \underset{\ell\times m}{T_{22}} \end{bmatrix} = \begin{bmatrix} \underset{(n-\ell)\times n}{T_1} \\ \underset{\ell\times n}{T_2} \end{bmatrix}$$

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$$T_2 \cdot x_t = [T_{21} \; T_{22}] \cdot \begin{bmatrix} z_t \\ p_t \end{bmatrix} = T_{21} z_t + T_{22} \cdot p_t$$

So for stability, I need

$$T_2 \cdot x_t + \sum \Lambda_2^{-1-j} T_2 B u_{t+j} = 0 \Leftrightarrow$$

$$\Leftrightarrow T_2 \cdot x_t = -\sum_{j=0}^{\infty} \Lambda_2^{-1} \cdot \Lambda_2^{-j} \cdot T_2 \cdot B \cdot u_{t+j} \Leftrightarrow$$

$$\Leftrightarrow T_2 \cdot x_t = -\Lambda_2^{-1} \cdot \sum_{j=0}^{\infty} \Lambda_2^{-j} \cdot T_2 \cdot B \cdot u_{t+j} \Leftrightarrow$$

$$\Leftrightarrow T_{21} z_t + T_{22} p_t = -\Lambda_2^{-1} \sum_{j=0}^{\infty} \Lambda_2^{-j} T_2 \cdot B \cdot u_{t+j} \Leftrightarrow$$

I don't know but I can choose

$$\Leftrightarrow T_{22} P_t = -T_{21} \cdot z_t - \Lambda_2^{-1} \sum_{j=0}^{\infty} \Lambda_2^{-j} T_2 \cdot B \cdot u_{t+j}$$

$$P_t = -(T_{22}^{-1} T_{21}) \cdot z_t - T_{22}^{-1} \Lambda_2^{-1} \sum_{j=0}^{\infty} \Lambda_2^{-j} T_2 \cdot B \cdot u_{t+j}$$

I need to be able to define T_{22}^{-1}

$$\underset{\ell\times m}{T_{22}} \cdot \underset{m\times1}{P_t}$$

$\ell = m$ for T_{22} to be a square matrix

For T_{22} to be a square matrix

$\ell = m$

Saddle path stability condition

unstable roots = # control (free, jump) variables

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$$y_t = N_1 \cdot y_{t-1} + N_2 \cdot y_{t-2} + N_3 \cdot y_{t-3} + N_4 \cdot y_{t-4} + \dots + N_k \cdot y_{t-k}$$

k-th order linear difference equation

equivalent to a system of k first order d.e.

Define (k-1) auxiliary variables

$$\left.\begin{aligned}
x1_t &\equiv y_{t-1} \\
x2_t &\equiv y_{t-2} = x1_{t-1} \\
x3_t &\equiv y_{t-3} = x2_{t-1} \\
&\text{and so on} \\
&\vdots \\
x_{k-1_t}&
\end{aligned}\right\}
\quad x_t = \begin{bmatrix} y_t \\ x1_t \\ x2_t \\ \vdots \\ x_{k-1_t} \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & \cdots & N_{k-1} & N_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_{t-1} \\ x1_{t-1} \\ x2_{t-1} \\ \vdots \\ x_{k-1_{t-1}} \end{bmatrix}$$

$$x_t = N \cdot x_{t-1}$$

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Phase Diagrams

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}}_{\text{eigenvalues}} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \overbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}^{b} \iff \begin{cases} \dot{x}_1 = x_1 + 2x_2 + 1 \\ \dot{x}_2 = -x_2 + 2 \end{cases}$$

eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -1$

eigenvectors: $\lambda_1 \to [1, 0]$, $\lambda_2 \to [1, -1]$

$\dot{x}_i \equiv \dfrac{dx_i}{dt}$

$\dot{x} = A \cdot x + b$

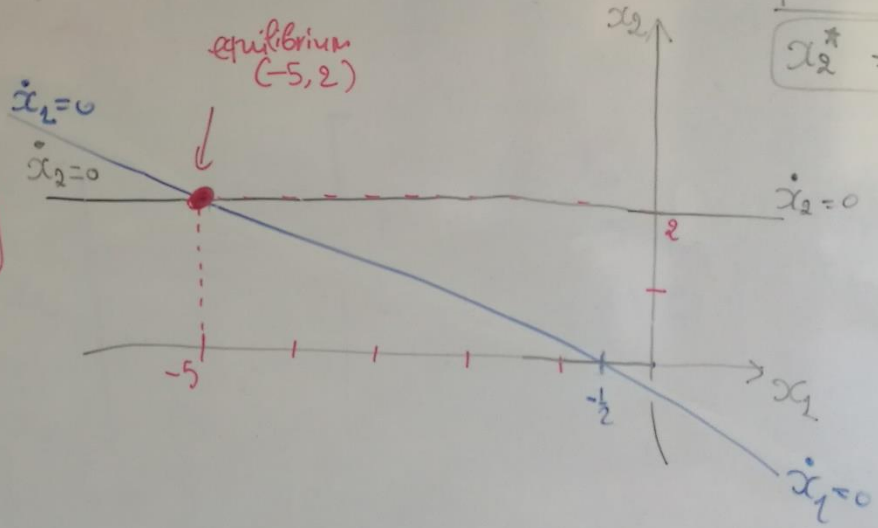
$\dot{x}_1 = 0 \iff x_1 + 2x_2 + 1 = 0 \iff x_2 = -\frac{1}{2} - \frac{1}{2} x_1$

$\dot{x}_2 = 0 \iff -x_2 + 2 = 0 \iff x_2 = 2$

Define the equilibrium point $\dot{x}_1 = \dot{x}_2 = 0$:

$2 = -\frac{1}{2} - \frac{1}{2} x_1 \iff \boxed{x_1^* = -5}$

$\boxed{x_2^* = 2}$



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Phase Diagrams

If I find myself in region: α, c

NEVER move towards $(-5, 2)$

b or d in general move away from equilibrium

unless

I am in some special position

$\dot{x}_1 = 0 \iff x_1 + 2x_2 + 1 = 0 \iff x_2 = -\frac{1}{2} - \frac{1}{2}x_1$

$\dot{x}_2 = 0 \iff -x_2 + 2 = 0 \iff x_2 = 2$

Define the equilibrium point $\dot{x}_1 = \dot{x}_2 = 0$:

$2 = -\frac{1}{2} - \frac{1}{2}x_1 \iff \boxed{x_1^* = -5}$

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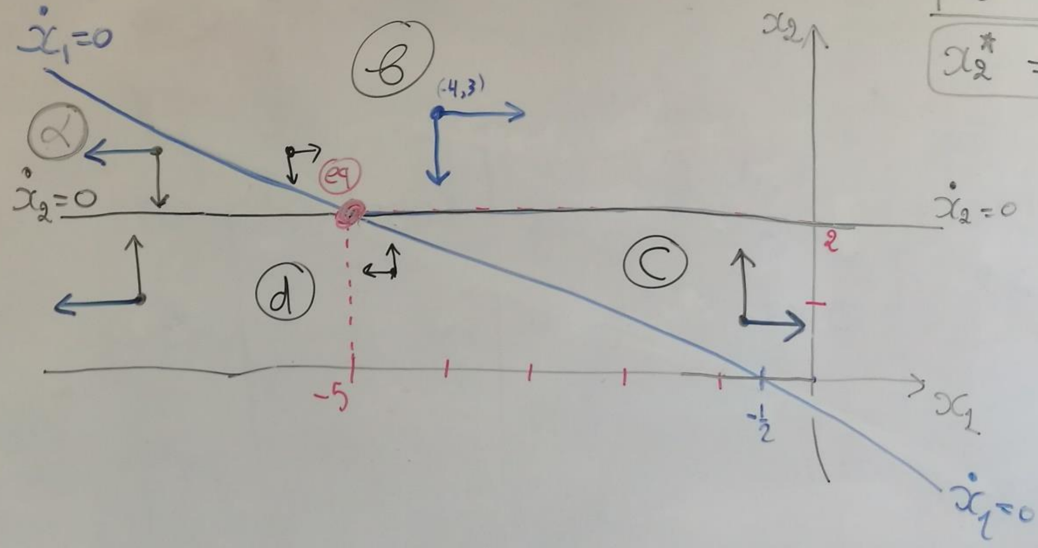

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$$\dot{x} = A \cdot x + b$$

$\lambda_s < 0$, v^s

$\lambda_u > 0$, v^u

eigenvalues eigenvectors

x^e particular solution $x^e = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$

Solution

$$x(t) = K_s \cdot v^s \cdot e^{\lambda_s t} + K_u \cdot v^u \cdot e^{\lambda_u t} + x^e$$

unstable unless $K_u = 0$ $\Big\}$ $x(t) = K_s \cdot v^s \cdot e^{\lambda_s \cdot t} + x^e \Leftrightarrow$

$$\Leftrightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = K_s \cdot \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \cdot e^{\lambda_s \cdot t} \Leftrightarrow \begin{aligned} x_1(t) - \bar{x}_1 &= K_s \cdot v_1^s \cdot e^{\lambda_s \cdot t} \\ x_2(t) - \bar{x}_2 &= K_s \cdot v_2^s \cdot e^{\lambda_s \cdot t} \end{aligned} \Bigg\} \boxed{\frac{x_1(t) - \bar{x}_1}{x_2(t) - \bar{x}_2} = \frac{v_1^s}{v_2^s}}$$

Set $K_s = 0$

unstable manifold

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ all other unstable paths = integral curves

$$\frac{x_1(t) - \bar{x}_1}{x_2(t) - \bar{x}_2} = \frac{v_1^s}{v_2^s}$$

$\lambda^s = -1$, $v^s = [1, -1]$

$\lambda^u = 1$, $v^u = [0, 1]$

Stable Manifold

If I start from a point $(\tilde{x}_1, \tilde{x}_2)$ on this line then my path will converge to x^e

In the example: $\boxed{\dfrac{x_1 + 5}{x_2 - 2} = -1}$

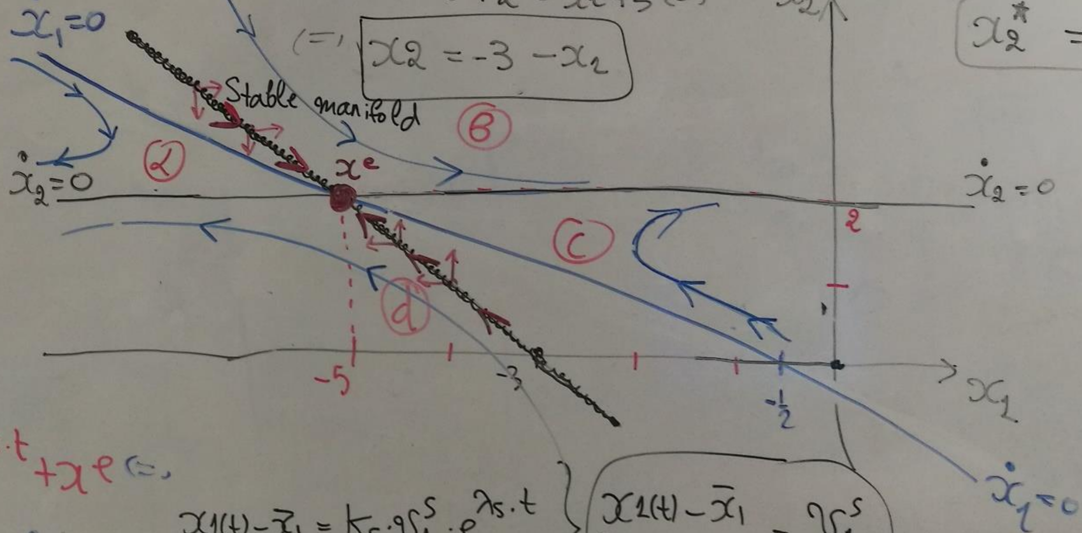
$\Leftrightarrow -(x_2 - 2) = x_1 + 5 \Leftrightarrow$

$\Leftrightarrow -x_2 + 2 = x_1 + 5 \Leftrightarrow$

$\Leftrightarrow \boxed{x_2 = -3 - x_1}$

$\boxed{x_1^* = -5}$

$\boxed{x_2^* = 2}$



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