Lecture 8

1

$$K_{t+1} - (1-\delta)K_t = I_t$$

Lag operator $\mathcal{L}$. (Forward Operator $\mathcal{F}$)

$$\mathcal{L} y_t \equiv y_{t-1}$$

$$\mathcal{L}^{-1} y_t \equiv y_{t-(-1)} = y_{t+1} \equiv \mathcal{F} y_t$$

$$\mathcal{L}^n y_t \equiv y_{t-n}$$

$$1 + \beta + \beta^2 + \beta^3 + \ldots = \frac{1}{1-\beta}$$

$$1 + \alpha \cdot \mathcal{L} + \alpha^2 \cdot \mathcal{L}^2 + \ldots = \sum_{i=0}^{\infty} \alpha^i \mathcal{L}^i =$$

$$= [1 - \alpha \mathcal{L}]^{-1}$$

$$[1-\alpha\mathcal{L}] \cdot [1-\alpha\mathcal{L}]^{-1} = 1 \Leftarrow,$$

$$(1 - \alpha\mathcal{L})[1 + \alpha \cdot \mathcal{L} + \alpha^2 \mathcal{L}^2 + \ldots] =$$

$$[1 + \alpha\mathcal{L} + \alpha^2\mathcal{L}^2 + \ldots] - [\alpha\mathcal{L} + \alpha^2\mathcal{L}^2 + \ldots] = 1 \checkmark$$

$$c_1 \cdot y_t + c_0 \cdot y_{t-1} = g(t)$$

$$g(t) \equiv X_t$$

$$\Leftrightarrow \frac{c_1}{c_1} \cdot y_t + \frac{c_0}{c_1} \cdot y_{t-1} = \frac{X_t}{c_1}$$

$$\beta \equiv \frac{c_0}{c_1}, \quad x_t \equiv \frac{X_t}{c_1}$$

$$y_t + \beta \cdot y_{t-1} = x_t$$

Particular Solution

$$y_t + \beta \cdot y_{t-1} = x_t \Leftrightarrow y_t + \beta \mathcal{L} y_t = x_t \Leftrightarrow$$

$$\Leftrightarrow [1 + \beta \cdot \mathcal{L}] y_t = x_t \Leftrightarrow$$

$$\Leftrightarrow y_t = [1 + \beta\mathcal{L}]^{-1} x_t \Leftarrow$$

$$\Leftrightarrow y_t = [1 - (-\beta)\mathcal{L}]^{-1} x_t = [1 + (-\beta)\mathcal{L} + (-\beta)^2 \mathcal{L}^2 + \ldots] x_t \Leftrightarrow y_t = \sum_{i=0}^{\infty} (-\beta)^i \mathcal{L}^i x_t = \sum_{i=0}^{\infty} (-\beta)^i x_{t-i}$$

Bounded iff $|-\beta| < 1$

Lecture 8

Lag operator: $\mathcal{L}^n y_t \equiv y_{t-n}$ $\quad y_t + \beta \cdot y_{t-1} = x_t$

Forward Operator

$\mathcal{F} \equiv \mathcal{L}^{-1}$

What if $|-\beta|>1$

$$1-\alpha\cdot\mathcal{L} = -\alpha\mathcal{L}\left[-(\alpha\mathcal{L})^{-1}+1\right] =$$
$$= -\alpha\mathcal{L}\cdot\left[1-\frac{1}{\alpha\mathcal{L}}\right]$$

$$\left[1-(\alpha\mathcal{L})^{-1}\right]^{-1} = 1+(\alpha\mathcal{L})^{-1}+(\alpha\mathcal{L})^{-2}+\dots =$$
$$= 1+\alpha^{-1}\mathcal{L}^{-1}+\alpha^{-2}\mathcal{L}^{-2}+\dots =$$
$$= 1+\alpha^{-1}\mathcal{F}+\alpha^{-2}\mathcal{F}^{2}+\dots$$

$$(1-\alpha\mathcal{L})^{-1} = \left\{-\alpha\mathcal{L}\cdot\left[1-(\alpha\mathcal{L})^{-1}\right]\right\}^{-1} =$$
$$= -(\alpha\mathcal{L})^{-1}\cdot\left[1-(\alpha\mathcal{L})^{-1}\right]^{-1} =$$
$$= -(\alpha)^{-1}\mathcal{L}^{-1}\cdot\left[1+\alpha^{-1}\mathcal{F}+\alpha^{-2}\mathcal{F}^{2}+\dots\right] =$$
$$= -\alpha^{-1}\mathcal{F}\cdot\left[1+\alpha^{-1}\mathcal{F}+(\alpha^{-1})^{2}\mathcal{F}^{2}+\dots\right] \Leftrightarrow$$
$$\Leftrightarrow \left[1-\alpha\mathcal{L}\right]^{-1} = \left[-(\alpha^{-1})\mathcal{F}-(\alpha^{-1})^{2}\mathcal{F}^{2}-(\alpha^{-1})^{3}\mathcal{F}^{3}+\dots\right. \Leftrightarrow \left[1-\alpha\mathcal{L}\right]^{-1} = -\sum_{i=1}^{\infty}(\alpha^{-1})^{i}\mathcal{F}^{i}$$

$|-\beta|<1$: $\bar{y}_t = \sum_{i=0}^{\infty}(-\beta)^{i}\cdot x_{t-i}$ [backward solution]

$|-\beta|>1$: $\bar{y}_t = -\sum_{i=1}^{\infty}(-\beta)^{-i}\mathcal{F}^{i}x_{t}$ [forward solution]

$$= -\sum_{i=1}^{\infty}\left(-\frac{1}{\beta}\right)^{i}\mathcal{F}^{i}x_t =$$
$$= -\sum_{i=1}^{\infty}\left(-\frac{1}{\beta}\right)^{i}x_{t+i}$$

General Solution of a non-homogeneous ODE =

= General solution of the homogeneous + Particular Solution

$$y_t = A\cdot(-\beta)^{t} + \bar{y}_t$$

$t=t^*$, $y_{t^*}=y^*$: $y_{t^*} = A\cdot(-\beta)^{t^*}+\bar{y}_{t^*}$

$$\Leftrightarrow A = \frac{y^*-\bar{y}_{t^*}}{(-\beta)^{t^*}}$$

Lecture 8

2nd Order ODE

$$C_2 \cdot y_t + C_1 \cdot y_{t-1} + C_0 \cdot y_t = g(t)$$

Homogeneous

$$\frac{C_2}{C_2} \cdot y_t + \frac{C_1}{C_2} \cdot y_{t-1} + \frac{C_0}{C_2} \cdot y_{t-2} = 0$$

$C_2 \neq 0$

$C_0 \neq 0$ $\quad \alpha_1 \equiv \frac{C_1}{C_2}$, $\alpha_2 \equiv \frac{C_0}{C_2}$

$$\left.\begin{array}{l} y_t + \alpha_1 \cdot y_{t-1} + \alpha_2 \cdot y_{t-2} = 0 \\ \text{Wise guess} \\ y_t = f(t) = \lambda^t \quad \begin{cases} y_{t-1} = \lambda^{t-1} \\ y_{t-2} = \lambda^{t-2} \end{cases} \end{array}\right\} \Leftarrow$$

$$\Leftarrow \lambda^t + \alpha_1 \cdot \lambda^{t-1} + \alpha_2 \cdot \lambda^{t-2} = 0 \Leftarrow$$

$$\Leftarrow \lambda^{t-2} \cdot [\lambda^2 + \alpha_1 \cdot \lambda + \alpha_2] = 0 \Leftarrow \boxed{\lambda^2 + \alpha_1 \cdot \lambda + \alpha_2 = 0}$$

Characteristic function

3

$$\lambda^2 + \alpha_1 \cdot \lambda + \alpha_2 = 0$$

$$\lambda_{1,2} = \frac{-\alpha_1 \pm \underbrace{\sqrt{\alpha_1^2 - 4\alpha_2}}_{\sqrt{\text{Discriminant } \Delta}}}{2} \qquad \Delta \equiv \alpha_1^2 - 4\alpha_2$$

I. $\Delta > 0$: λ_1, λ_2 Real and Distinct : $y_t = A_1 \cdot \lambda_1^t + A_2 \cdot \lambda_2^t$

λ_1^t and λ_2^t solutions $\Leftarrow$

$\Leftarrow y_t = A_1 \cdot \lambda_1^t + A_2 \cdot \lambda_2^t$ also a solution

$$\left.\begin{array}{l} t = t^*, \; y_{t^*} = y^* \\ t = t^{**}, \; y_{t^{**}} = y^{**} \end{array}\right\} \begin{array}{l} y^* = A_1 \cdot \lambda_1^{t^*} + A_2 \cdot \lambda_2^{t^*} \\ y^{**} = A_1 \cdot \lambda_1^{t^{**}} + A_2 \cdot \lambda_2^{t^{**}} \end{array}$$

2 equations in two unknowns A_1, A_2

Stability ?

Equivalent : $\underbrace{\max\{|\lambda_1|, |\lambda_2|\}}_{\text{Dominant Root}} < 1$

Stability requires : $|\lambda_1| < 1$ and $|\lambda_2| < 1$

II. $\Delta = 0$: Two real and equal roots : $\lambda_{1,2} = \lambda^* = -\frac{\alpha_1}{2}$

(Multiple root)

λ^{*t} is a solution

$t \cdot \lambda^{*t}$ is a solution [check]

$$y_t = A_1 \cdot \lambda^{*t} + A_2 \cdot t \cdot \lambda^{*t} =$$

$$= [A_1 + A_2 \cdot t] \cdot \lambda^{*t}$$

Stability : $|\lambda^*| < 1$

Lecture 8

4

2nd Order ODE

$c_2 \cdot y_t + c_1 \cdot y_{t-1} + c_0 \cdot y_t = g(t)$

Homogeneous

$\frac{c_2}{c_2} \cdot y_t + \frac{c_1}{c_2} \cdot y_{t-1} + \frac{c_0}{c_2} \cdot y_{t-2} = 0$

$c_2 \neq 0$

$c_0 \neq 0$ $\quad \alpha_1 \equiv \frac{c_1}{c_2}$, $\alpha_2 \equiv \frac{c_0}{c_2}$

$y_t + \alpha_1 \cdot y_{t-1} + \alpha_2 \cdot y_{t-2} = 0$ $\Leftarrow,$

$\lambda^2 + \alpha_1 \cdot \lambda + \alpha_2 = 0$

Discriminant Δ

$\lambda_{1,2} = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_2}}{2}$ $\qquad \Delta \equiv \alpha_1^2 - 4\alpha_2$

I. $\Delta > 0$: λ_1, λ_2 Real and Distinct : $y_t = A_1 \cdot \lambda_1^t + A_2 \cdot \lambda_2^t$

λ_1^t and λ_2^t solutions $\Leftarrow,$

$\Leftarrow, y_t = A_1 \cdot \lambda_1^t + A_2 \cdot \lambda_2^t$ also a solution

$t = t^*$, $y_{t^*} = y^*$ $\quad y^* = A_1 \cdot \lambda_1^{t^*} + A_2 \cdot \lambda_2^{t^*}$

$t = t^{**}$, $y_{t^{**}} = y^{**}$ $\quad y^{**} = A_1 \cdot \lambda_1^{t^{**}} + A_2 \cdot \lambda_2^{t^{**}}$

2 equations in two unknowns A_1, A_2

Stability ?

Stability requires : $|\lambda_1| < 1$ and $|\lambda_2| < 1$

Equivalent : $\underbrace{\max\{|\lambda_1|, |\lambda_2|\}}_{\text{Dominant Root}} < 1$

II. $\Delta = 0$: Two real and equal roots : $\lambda_{1,2} = \lambda^* = -\frac{\alpha_1}{2}$

(Multiple root)

λ^{*t} is a solution

$t \cdot \lambda^{*t}$ is a solution [check]

$y_t = A_1 \cdot \lambda^{*t} + A_2 \cdot t \cdot \lambda^{*t} =$

$= [A_1 + A_2 \cdot t] \cdot \lambda^{*t}$

Stability : $|\lambda^*| < 1$

III. $\Delta < 0$, λ_1, λ_2 are Two Complex conjugate numbers

$\lambda_{1,2} = \underbrace{\alpha}_{\text{real part}} \pm \underbrace{i \cdot \theta}_{\text{Imaginary Part}}$, $i = \sqrt{-1}$

$\alpha, \theta \in \mathbb{R}$

$\alpha = -\frac{1}{2}\alpha_1$

$\theta = \frac{1}{2}\sqrt{4\alpha_2 - \alpha_1^2}$

$y_t = A' \cdot (\alpha + i\theta)^t + A'' \cdot (\alpha - i\theta)^t$

Polar form of complex numbers

(De Moivre Theorem)

Stability requires $|r| < 1$ $\quad y_t = r^t \cdot [A_1 \cdot \cos wt + A_2 \cdot \sin wt]$

Lecture 8

2^{nd} Order ODE

$C_2 \cdot y_t + C_1 \cdot y_{t-1} + C_0 \cdot y_t = g(t)$

Particular Solution

$\bar{y}_t \to$ like $g(t)$

$g(t) = G$
Guess

$\bar{y}_t = B$

$\to C_2 \cdot B + C_1 \cdot B + C_0 \cdot B = G \Leftrightarrow$
(Method of undetermined coefficients)

$\Leftrightarrow B = \frac{G}{C_2+C_1+C_0}$, iff $C_0+C_1+C_2 \neq 0$

If $C_0+C_1+C_2 = 0$

New Guess: $\bar{y}_t = B \cdot t$ $y_{t-1} = B(t-1)$ $y_{t-2} = B\cdot(t-2)$

$C_2 \cdot B \cdot t + C_1 \cdot B \cdot (t-1) + C_0 \cdot B(t-2) = G \Leftrightarrow$

$\Leftrightarrow (C_2+C_1+C_0) B \cdot t - B \cdot C_1 - B \cdot 2C_0 = G \Leftrightarrow B = \frac{-G}{C_1+2C_0}$, iff $C_1+2C_0 \neq 0$

If $C_1+2C_0 = 0$, New Guess : $\bar{y}_t = B \cdot t^2$ ... CHECK $\to$ A real solution

$g(t) \to$ Generic function of time

$\frac{C_2 \cdot y_t}{C_2} + \frac{C_1 \cdot y_{t-1}}{C_2} + \frac{C_0 \cdot y_t}{C_2} = \frac{X_t}{C_2} \equiv x_t$

$y_t + \alpha_1 \cdot y_{t-1} + \alpha_2 \cdot y_t = x_t$

In Operators notation :

$y_t + \alpha_1 \cdot L y_t + \alpha_2 L^2 y_t = x_t \Leftrightarrow$

$\Leftrightarrow [1 + \alpha_1 L + \alpha_2 L^2] y_t = x_t \Leftrightarrow$

$\Leftrightarrow y_t = [1 + \alpha_1 L + \alpha_2 L^2]^{-1} \cdot x_t$

$1 + \alpha_1 L + \alpha_2 L^2 = L^2 \cdot [L^{-2} + \alpha_1 L^{-1} + \alpha_2] =$

$= F^{-2} \cdot [F^2 + \alpha_1 F + \alpha_2]$

2nd order polynomial
two roots φ_1 φ_2

Characteristic function

$[F^2 + \alpha_1 F + \alpha_2] \underset{\text{factorise}}{=} (F - \varphi_1) \cdot (F - \varphi_2)$

$1 + \alpha_1 L + \alpha_2 L^2 = L^2 \cdot [F - \lambda_1] \cdot [F - \lambda_2]$

φ_1, φ_2 concide with λ_1, λ_2

Lecture 8

2nd Order ODE

(★) $1+\alpha_1 L+\alpha_2 L^2 = L^2\cdot(F-\lambda_1)(F-\lambda_2) =$

$=(LF-\lambda_1 L)\cdot(LF-\lambda_2 L) =$

$=\{F \equiv L^{-1}\} =$

$=(1-\lambda_1 L)(1-\lambda_2 L)$

Thus the Particular Solution is:
[two real distinct roots λ_1, λ_2]

$$y_t = \frac{1}{[1+\alpha_1 L+\alpha_2 L^2]}\, x_t$$

$$= \frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)}\, x_t = \left\{\begin{array}{c}\text{Determine constants}\\ \theta_1, \theta_2\end{array}\right\} =$$

$\theta_1 = \frac{\lambda_1}{\lambda_1-\lambda_2}$, $\theta_2 = \frac{-\lambda_2}{\lambda_1-\lambda_2}$

$$= \left[\frac{\theta_1}{1-\lambda_1 L} + \frac{\theta_2}{1-\lambda_2 L}\right] x_t$$

$g(t) \rightarrow$ Generic function of time

$$\frac{c_2\cdot y_t}{c_2} + \frac{c_1\cdot y_{t-1}}{c_2} + \frac{c_0\cdot y_t}{c_2} = \frac{X_t}{c_2} \equiv x_t$$

$y_t + \alpha_1\cdot y_{t-1} + \alpha_2\cdot y_t = x_t$

In Operators notation:

$y_t + \alpha_1\cdot L y_t + \alpha_2 L^2 y_t = x_t \iff$

$\iff [1+\alpha_1 L+\alpha_2 L^2]\, y_t = x_t \iff$

$\iff y_t = [1+\alpha_1 L+\alpha_2 L^2]^{-1}\cdot x_t$

$1+\alpha_1\cdot L+\alpha_2 L^2 = L^2\cdot[L^{-2}+\alpha_1\cdot L^{-1}+\alpha_2] =$

$= F^2\cdot[F^2+\alpha_1 F+\alpha_2]$ → Characteristic function

2nd order polynomial
two roots φ_1 φ_2

$[F^2+\alpha_1 F+\alpha_2] \underset{\text{factorise}}{=} (F-\varphi_1)\cdot(F-\varphi_2)$

φ_1, φ_2 coincide with λ_1, λ_2

$1+\alpha_1\cdot L+\alpha_2 L^2 = L^2\cdot[F-\lambda_1]\cdot[F-\lambda_2] \iff$ (★)

Assumption

Two distinct real roots λ_1, λ_2 characteristic function