Lecture 6

Nonlinear function $f(x,y)$ (1)

Linearization (First Order / Taylor)

↳ Linear approximation around a point (x^*, y^*)

$$f(x,y) \simeq f(x,y)\Big|_{x^*,y^*} + \frac{\partial f(x,y)}{\partial x}\Big|_{x^*,y^*} \cdot (x - x^*) + \frac{\partial f(x,y)}{\partial y}\Big|_{x^*,y^*} \cdot (y - y^*)$$

Constants

Log-linearization [with respect to the logs of the variables)

$x \equiv e^{\ln x}$, $y \equiv e^{\ln y}$, $f(x,y) \equiv f(e^{\ln x}, e^{\ln y})$

$$f(x,y) \simeq f(x,y)\Big|_{x^*,y^*} + \frac{\partial f(x,y)}{\ln x}\Big|_{x^*,y^*} \cdot (\ln x - \ln x^*) + \frac{\partial f(x,y)}{\partial \ln y}\Big|_{x^*,y^*} \cdot (\ln y - \ln y^*)$$

Lecture 6 — Discrete Time: Difference Equations (2)

[Gandolfo]

$y_t = f(t)$

(distance is h)

$$\Delta y_t = f(t+h) - f(t)$$

First Difference: y_{t+h}, y_t

$t, t-h$

$\boxed{h = 1}$ one unit of time

$\Delta^n y_t$

Δy_t			1		-1			
$\Delta^2 y_t$		1		-2		1		
$\Delta^3 y_t$	1		-3		3		-1	
	1		-4		? 6		-4	1 $\Delta^4 y$

Pattern of coefficients: Pascal Triangle

$\Delta y_t = f(t+1) - f(t) = y_{t+1} - y_t$

$\Delta y_{t+1} = f(t+2) - f(t+1) = y_{t+2} - y_{t+1}$

$\vdots$

Second difference

$$\Delta^2 y_t = \Delta(\Delta y_t) = \Delta\left(y_{t+1} - y_t\right) =$$
$$= \Delta y_{t+1} - \Delta y_t =$$
$$= y_{t+2} - y_{t+1} - [y_{t+1} - y_t] =$$
$$= y_{t+2} - 2\cdot y_{t+1} + y_t$$

And So on ...

Third Difference $\Delta^3 y_t = \Delta(\Delta^2 y_t) =$

$$= \Delta\left(y_{t+2} - 2y_{t+1} + y_t\right) = \Delta y_{t+2} - 2\Delta y_{t+1} + \Delta y_t =$$
$$= y_{t+3} - y_{t+2} - 2(y_{t+2} - y_{t+1}) + y_{t+1} - y_t \Leftrightarrow$$
$$\Leftrightarrow \Delta^3 y_t = y_{t+3} - 3y_{t+2} + 3y_{t+1} - y_t$$

Lecture 6

Discrete Time: Difference Equations

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[Gandolfo]

$\boxed{\Delta^n y_t}$, $\Delta y_t \equiv y_{t+1} - y_t$

A function involving differences of y_t ($n=1$ or higher)

$\hookrightarrow y_t = f(t) \longrightarrow$ unknown function

- ODE (ordinary difference equation)

Because y is a function of time only

$y = f(t)$

[y is a function of time / a function I don't know]

- Order of the difference equation

[partial difference equation

$y = f(t, x_1, x_2, \dots)$]

$\boxed{\text{Vintage}}$

vintage capital models

t

vintage (t^*)

($t - t^*$)

$$K_{t+1} = (1-\delta)K_t + I_t \Leftarrow,$$

$$\Leftarrow, \quad K_{t+1} = K_t - \delta K_t + I_t \Leftarrow,$$

$$\Leftarrow, \quad K_{t+1} - K_t + \delta K_t + I_t = 0$$

Ordinary Difference Equation

I_t : time series

$I_t = I(t)$

Lecture 6 Discrete Time: Difference Equations

[Gandolfo]

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$$\alpha y_{t+5} + \beta y_{t+2} + \gamma y_t = 0 \rightarrow \text{5th order}$$

Solution → Look for the (unknown) function of time that will give $y_t = f(t)$

$x^2 - \alpha = 0$

Difference equation → A functional equation involving $y_t, y_{t+1}, \ldots$ where $y = f(t)$

$\Delta y_t = \alpha \Leftarrow,$

$\Leftarrow, \boxed{y_{t+1} - y_t = \alpha}$

$y_t - y_{t-1} = \alpha$

$y_{t-345} - y_{t-346}$

Guess $y_t = f(t) = \alpha \cdot t$ (f unknown)

$y_{t+1} = f(t+1) = \alpha \cdot (t+1)$

$\alpha \cdot (t+1) - \alpha \cdot t = \cancel{\alpha \cdot t} + \alpha - \cancel{\alpha \cdot t} = \alpha$ ✓

Lecture 6

$\Delta y_t = \alpha \Leftrightarrow y_{t+1} - y_t = \alpha$ 5

Guess: $y_t = f(t) = \alpha \cdot t$
$y_{t+1} = f(t+1) = \alpha \cdot (t+1)$
$\Big\} \; \alpha \cdot (t+1) - \alpha \cdot t = \alpha \Leftrightarrow$
$\Leftrightarrow \alpha \cdot t + \alpha - \alpha \cdot t = \alpha \Leftrightarrow$
$\Leftrightarrow \alpha = \alpha$ ✓

$y_t = f(t) = \alpha \cdot t + \beta$
$y_{t+1} = f(t+1) = \alpha \cdot (t+1) + \beta$
1st order $\longrightarrow$ 1 arbitrary constant

$\Big\} \; \alpha \cdot (t+1) + \beta - [\alpha \cdot t + \beta] = \alpha \Leftrightarrow$
$\Leftrightarrow \cancel{\alpha \cdot t} + \alpha + \cancel{\beta} - \cancel{\alpha \cdot t} - \cancel{\beta} = \alpha$
$\Leftrightarrow \alpha = \alpha$ ✓

$y_t = \alpha \cdot t + \beta$
Family of functions $\beta \in \mathbb{R}$
arbitrary constant

Theorem 1
n^{th} order ODE
function of time
involving n arbitrary constants

All linear functions ($\beta \in \mathbb{R}$) with slope α solve the difference equation

$\Delta^2 y_t = 0 \Leftrightarrow$
$\Leftrightarrow y_{t+2} - 2y_{t+1} + y_t = 0$

Guess: $\begin{cases} y_t = a \cdot t + \beta \\ y_{t+1} = a \cdot (t+1) + \beta \\ y_{t+2} = a \cdot (t+2) + \beta \end{cases}$ ✓ For any arbitrary value of a, β

2nd order $\longrightarrow$ 2 arbitrary constants

Lecture 6 Linear ODE with constat coefficients

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n'th order / Non-Homogeneous

Forward formulation

$$C_n \cdot y_{t+n} + C_{n-1} \cdot y_{t+n-1} + \dots + C_1 \cdot y_{t+1} + C_0 \cdot y_t = g(t)$$

$$C_n \cdot y_t + C_{n-1} \cdot y_{t-1} + \dots + C_1 \cdot y_{t-n+1} + C_0 \cdot y_{t-n} = g(t)$$

Backward formulation

C_i, $i = 0, 1, \dots, n$ Constants , $C_0 \neq 0$, $C_n \neq 0$

$g(t)$ a known function of time

Some theorems on the solution of an n'th order Linear homogeneous ODE with constant coefficients

#1 If $y_1(t)$ is a solution then $A \cdot y_1(t)$ is also a solution

Homogeneous Equation

$$C_n \cdot y_{t+n} + C_{n-1} \cdot y_{t+n-1} + \dots + C_1 \cdot y_{t+1} + C_0 \cdot y_t = 0$$

$$C_n \cdot y_1(t+n) + C_{n-1} \cdot y_1(t+n-1) + \dots + C_0 \cdot y_1(t) = 0 \checkmark$$

$$A \cdot [C_n \cdot y_1(t+n) + C_{n-1} \cdot y_1(t+n-1) + \dots + C_0 \cdot y_1(t)] = 0$$

$$= C_n \cdot A \cdot y_1(t+n) + C_{n-1} \cdot A \cdot y_1(t+n-1) + \dots + C_0 \cdot A \cdot y_1(t) = 0 \checkmark$$

Lecture 6 Linear ODE with constat coefficients

n' th order / Non-Homogeneous

$$C_n \cdot y_{t+n} + C_{n-1} \cdot y_{t+n-1} + \dots + C_1 \cdot y_{t+1} + C_0 \cdot y_t = 0$$

Linear Independence of functions

Let n functions of time: $y_1(t)$, $y_2(t)$, ..., $y_n(t)$

Then $\longrightarrow$ linearly independent $A_i = 0, \forall i = 1, 2, \dots, n$

$[A_1, A_2, \dots, A_n]$

$$A_1 \cdot y_1(t) + A_2 \cdot y_2(t) + \dots + A_n \cdot y_n(t) = 0$$

[if $\exists\, i$, where $A_i \neq 0$ then $\longrightarrow$ linearly dependent]

Lecture 6 Linear ODE with constat coefficients

n'th order / Non-Homogeneous

$$C_n \cdot y_{t+n} + C_{n-1} \cdot y_{t+n-1} + \dots + C_1 \cdot y_{t+1} + C_0 \cdot y_t = 0$$

Theorem #3 If $y_1(t)$, $y_2(t)$ are linearly indepent solutions to the homogeneous ODE then $A_1 \cdot y_1(t) + A_2 \cdot y_2(t)$ is also a solution

Theorem #4 General Solution the homogeneous linear ODE with constant coefficients is $f(t; A_1, A_2, \dots, A_n) =$

$$= A_1 \cdot y_1(t) + A_2 \cdot y_2(t) + \dots + A_n \cdot y_n(t)$$

where $y_i(t)$, $i = 1, 2, \dots, n$ linearly independent functions that solve the homogeneous and A_i, $i = 1, \dots, n$ are n arbitrary constants

Lecture 6 Linear ODE with constat coefficients

Linear Homogeneous System
$\Downarrow$
Unique Solution
$|D(t)| \neq 0$

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n'th order / Non-Homogeneous

$$C_n \cdot y_{t+n} + C_{n-1} \cdot y_{t+n-1} + \dots + C_1 \cdot y_{t+1} + C_0 \cdot y_t = 0$$

Linear Independence $\forall t$

$$A_1 \cdot y_1(t) + \dots + A_n \cdot y_n(t) = 0$$

$$A_i = 0, \ \forall i, \ i=1,\dots,n$$

Theorem #5

Let $y_1(t), y_2(t), \dots, y_n(t)$ solutions to the homogeneous. These functions are linearly independent iff the following determinant is $\neq 0$:

$$D(t) = \begin{vmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \\ y_1(t+1) & y_2(t+1) & \dots & y_n(t+1) \\ \vdots & & & \\ y_1(t+n-1) & y_2(t+n-1) & \dots & y_n(t+n-1) \end{vmatrix}$$

Casorati Determinant

Linear Homogeneous System

$$A_1 \cdot y_1(t) + \dots + A_n \cdot y_n(t) = 0$$
$$A_1 \cdot y_1(t+1) + \dots + A_n \cdot y_n(t+1) = 0$$
$$\vdots$$
$$A_1 \cdot y_1(t+n-1) + \dots + A_n \cdot y_n(t+n-1) = 0$$

$\neq 0$ $D(t)$

$$\begin{bmatrix} y_1(t) & \dots & y_n(t) \\ \vdots & & \\ y_1(t+n-1) & \dots & y_n(t+n-1) \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$