Lecture 2

Continuous Time Dynamic Optimization Problem

$$\max_{\{x,u\}} \int_0^\infty \boxed{e^{-r\cdot t} \cdot f(x,u)}\, dt$$

s.t λ or m : $x' = \boxed{g(x,u)}$ $\left[x' \equiv \frac{dx}{dt}\right]$

$x(0)$ is known (initial condition)

$x \rightarrow$ state

$u \rightarrow$ control

$\lambda \rightarrow$ multiplier (present value) (costate variable)

or

$m \rightarrow$ (Current Value multiplier)

I. Optimal Control [Hamiltonian] (1)

Present Value Hamiltonian : $\boxed{H = e^{-r\cdot t} \cdot f(x,u) + \lambda \cdot g(x,u)}$

or

Current " " :

General Definition of the Hamiltonian

$$\boxed{\bar{H} = f(x,u) + m \cdot g(x,u)}$$

$$\max \int_0^\infty F(x,u)\, dt$$

s.t $x' = g(x,u)$

$$H = F(x,u) + \lambda \cdot g(x,u)$$

$$\bar{H} = e^{r\cdot t} \cdot H =$$

$$= e^{r\cdot t} \cdot \left[e^{-r\cdot t} \cdot f(x,u) + \lambda \cdot g(x,u)\right] =$$

$$= f(x,u) + \boxed{e^{r\cdot t} \cdot \lambda} \cdot g(x,u)$$

$$m \equiv e^{r\cdot t} \cdot \lambda$$

FOC (CV Hamiltonian)

$$\bar{H} = f(x,u) + m \cdot g(x,u)$$

(1) $\boxed{\frac{\partial \bar{H}}{\partial u} = 0} \Leftarrow, \quad \frac{\partial f(x,u)}{\partial u} + m \cdot \frac{\partial g(x,u)}{\partial u} = 0$

(2) $\boxed{m' = r \cdot m - \frac{\partial \bar{H}}{\partial x}} \Leftarrow, \quad m' = r \cdot m - \left[\frac{\partial f(x,u)}{\partial x} + m \cdot \frac{\partial g(x,u)}{\partial x}\right]$

(3) $\boxed{\frac{\partial \bar{H}}{\partial m} = x'} \Leftarrow, \quad x' = g(x,u)$ (the constraint)

Solve a dynamic system wrt x, u, m

Lecture 2

Continuous Time Dynamic Optimization Problem

$$\max_{\{x,u\}} \int_0^{\infty} \boxed{e^{-r\cdot t}\cdot f(x,u)}\, dt$$

s.t $\quad x' = \boxed{g(x,u)} \quad [x' \equiv \frac{dx}{dt}]$

$x(0)$ is Known (initial condition)

$x \rightarrow$ state

$u \rightarrow$ control

II. Dynamic Programming (Bellman Equation) 2

Hamilton – Jacobi – Bellman Equation

Define $J(t_0, x_0)$ the optimal value function starting at time t_0, in state x_0:

$$J(t_0, x_0) \equiv \max \int_{t_0}^{\infty} e^{-r\cdot t}\cdot f(x,u)\, dt$$

s.t $\quad x' = g(x,u)$

$x(t_0) = x(0)$, Known

$$J(t_0, x_0) \equiv \max \int_{t_0}^{\infty} e^{-r\cdot t + r\cdot t_0 - r\cdot t_0}\cdot f(x,u)\, dt$$

$$\Leftrightarrow J(t_0, x_0) \equiv \max e^{-r\cdot t_0}\cdot \underbrace{\int_{t_0}^{\infty} e^{-r\cdot(t-t_0)}\cdot f(x,u)\, dt}_{V(x_0)}$$

Define $V(x_0) \equiv \max \int_{t_0}^{\infty} e^{-r\cdot(t-t_0)}\cdot f(x,u)\, dt$

$$J(t_0, x_0) \equiv e^{-r\cdot t_0} V(x_0)$$

$$\frac{\partial J(t_0, x_0)}{\partial t} = -r\cdot e^{-r\cdot t_0} V(x_0)$$

$$\frac{\partial J(t_0, x_0)}{\partial x} = e^{-r t_0}\cdot V'(x_0)$$

It can be shown that $-\frac{\partial J}{\partial t} \simeq \max_u \left[e^{-r\cdot t}\cdot f(x,u) + e^{-rt}\cdot V'(x)\cdot g(x,u)\right]$

$$\Leftrightarrow -e^{r\cdot t}\cdot \frac{\partial J}{\partial t} = \boxed{r\cdot V(x) = \max_u \left[f(x,u) + V'(x)\cdot g(x,u)\right]}$$

Lecture 2

II. Dynamic Programming (Bellman Equation) ③

Hamilton-Jacobi-Bellman equation

$$r \cdot V(x) = \max_u \left[f(x,u) + V'(x) \cdot g(x,u) \right]$$

F.O.C.

$$u: \quad 0 = \frac{\partial f(x,u)}{\partial u} + V'(x) \cdot \frac{\partial g(x,u)}{\partial u}$$

Envelope or Benveniste-Scheinkman Condition:

$$r \cdot V'(x) = \frac{\partial f(x,u)}{\partial x} + V''(x) \cdot g(x,u) + V'(x) \cdot \frac{\partial g(x,u)}{\partial x}$$

$m \equiv V'(x)$

+ constraint $x' = g(x,u)$

Define $J(t_0, x_0)$ the optimal value function starting at time t_0, in state x_0:

★ Hamilton-Jacobi-Bellman Equation

$$J(t_0, x_0) \equiv \max \int_{t_0}^{\infty} e^{-r \cdot t} \cdot f(x,u)\, dt$$

s.t. $x' = g(x,u)$

$x(t_0) = x(0)$, Known

$$J(t_0, x_0) \equiv \max \int_{t_0}^{\infty} e^{-r \cdot t + r \cdot t_0 - r \cdot t_0} \cdot f(x,u)\, dt =$$

$$\Leftarrow J(t_0, x_0) \equiv \max e^{-r \cdot t_0} \cdot \underbrace{\int_{t_0}^{\infty} e^{-r \cdot (t - t_0)} \cdot f(x,u)\, dt}_{V(x_0)}$$

Define $V(x_0) \equiv \max \int_{t_0} e^{-r \cdot (t - t_0)} \cdot f(x,u)\, dt$

$$J(t_0, x_0) \equiv e^{-r \cdot t_0} V(x_0)$$

$$\frac{\partial J(t_0, x_0)}{\partial t} = -r \cdot e^{-r \cdot t_0} V(x_0)$$

$$\frac{\partial J(t_0, x_0)}{\partial x} = e^{-r t_0} \cdot V'(x_0)$$

It can be shown that $-\frac{\partial J}{\partial t} \simeq \max_u \left[e^{-r \cdot t} f(x,u) + e^{-rt} \cdot V'(x) \cdot g(x,u) \right]$

$$\Leftarrow -e^{r \cdot t} \frac{\partial J}{\partial t} = r \cdot V(x) = \max_u \left[f(x,u) + V'(x) \cdot g(x,u) \right] \quad ★$$

Lecture 2

Optimal Control

[Pontryagin]

General Optimal Control Problem

$T \to$ Time Horizon

$$\max_{u} \int_0^T f(t, x, u)\, dt$$

s.t $x' = g(t, x, u)$

$x(0) = x_0$ initial condition

and/or

$x(T) \geq x_T$ terminal condition

$x \to$ State variables

$u \to$ control "

Consumer Model

↳ Lives $\to [0, T]$

Stock of savings $S(t)$ at time t

Some of $S(t)$ can be consumed $c(t)$

The remaining is saved, gaining a rate r

Thus $\to$ Budget Constraint

$$S'(t) = r \cdot S(t) - c(t)$$

$$\boxed{S' = r \cdot S - c}$$

Initial Savings S_0

temporal utility

Lifetime utility

$$\max_{\{c\}} \int_0^T e^{-\rho \cdot t} \cdot u(c)\, dt$$

$S' = r \cdot S - c$

initial endowment $\to$ $S(0) = S_0$

$S(T) \geq 0 \to$ sustainability condition [cannot owe in T]

ρ the rate of time preference

Lecture 2

Sketch of the Solution Method (5)

Optimal Control
[Pontryagin]

General Optimal Control Problem

T → Time Horizon

$$\max_{u} \int_0^T f(t,x,u)\,dt$$

s.t $x' = g(t,x,u)$

$x(0) = x_0$ initial condition
and/or
$x(T) \geq \bar{x}$ terminal condition

$x \to$ State variables
$u \to$ control "

Constrained Optimization problem
↓
1) Terminal Condition (1)
2) Dynamic constraint (∞)
(differential equation)

$$\int_0^T \lambda \cdot g(t,x,u)\cdot dt \; - \int_0^T \lambda \cdot x' \, dt$$

$$\mathcal{L} = \int_0^T f(t,x,u)\,dt + \int_0^T \lambda \cdot [g(t,x,u) - x']\,dt + \mu \cdot [x(T) - \bar{x}] \quad (1)$$

multiplier

co-state variables

Integrate the term $-\int_0^T \lambda \cdot x'\,dt$ in $\mathcal{L}$ by parts

$$-\int_0^T \lambda \cdot x'\,dt = -[\lambda(T)\cdot x(T) - \lambda(0)\cdot x(0)] + \int_0^T \lambda' \cdot x\,dt \quad (2)$$

Substitute (2) in (1)

Lecture 2

Optimal Control

Observe that

If I define

$$\mathcal{H} = f(t,x,u) + \lambda \cdot g(t,x,u)$$

Hamiltonian

the FOC 3 + 4

can be written in term of derivatives of $\mathcal{H}$:

3 $\rightarrow \dfrac{\partial \mathcal{H}}{\partial u} = 0$

4 $\rightarrow \dfrac{\partial \mathcal{H}}{\partial x} = -\lambda'$

$$x' = \frac{\partial \mathcal{H}}{\partial \lambda}$$

$$\lambda(T) \cdot \{x(T) - \bar{x}\} = 0$$

$\rightarrow$ Kuhn-Tucker complementary slackness condition

Sketch of the Solution Method 6

2 $\xrightarrow{\text{in}}$ 1

$$\mathcal{L} = \int_0^T f(t,x,u)\,dt + \int_0^T \lambda \cdot g(t,x,u) - [\lambda(T) \cdot x(T) - \lambda(0) \cdot x(0)] + $$

$$+ \int_0^T \lambda' \cdot x\,dt + \mu \cdot [x(T) - \bar{x}]$$

$$\Leftrightarrow \mathcal{L} = \int_0^T \{ f(t,x,u) + \lambda \cdot g(t,x,u) + \lambda' \cdot x \}\,dt +$$

$$+ \mu \cdot [x(T) - \bar{x}] - [\lambda(T) \cdot x(T) - \lambda(0) \cdot x(0)]$$

To maximize $\mathcal{L}$ by choosing x, u all it is necessary to do is to maximize $\{ f(t,x,u) + \lambda \cdot g(t,x,u) + \lambda' x \}$

FOC $\rightarrow$ $\{\}$

$$\frac{\partial\{\}}{\partial u} = 0 \Leftrightarrow \frac{\partial f}{\partial u} + \lambda \cdot \frac{\partial g}{\partial u} = 0, \ \forall t \quad 3$$

$$\frac{\partial\{\}}{\partial x} = 0 \quad \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} + \lambda' = 0, \ \forall t \quad 4$$

+ Constraints. + $\dfrac{\partial\{\}}{\partial \mu} = 0$ + complementary slackness condition