MSc MATHS ECON - Tutorial 5

Qualitative Analysis

Spyros Tsangaris, Maria Gioka

Athens University of Economics and Business

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Non-autonomous System

The system of n first-order differential equations can be written as:

$$\left\{ \begin{array}{c} \dot{x}_1 = f_1(t, x_1(t), x_2(t), \ldots, x_n(t)) \\ \dot{x}_2 = f_2(t, x_1(t), x_2(t), \ldots, x_n(t)) \\ \vdots \\ \dot{x}_n = f_n(t, x_1(t), x_2(t), \ldots, x_n(t)) \end{array} \right\}$$

where $\dot{x}(t) = \frac{dx}{dt}$. In vector notation:

$$\dot{x}(t) = f(x(t), t)$$

Qualitative analysis analyzes differential equations without solving them analytically or numerically. Therefore, we can obtain the behavior of the solution without having them explicitly.

Autonomous System

When f does not explicitly depend on time the system is called autonomous

$$\dot{x}(t) = f(x(t))$$

The $n = 1$ case

$$\dot{x}_1 = f(x_1(t))$$

The $n = 2$ case

$$\left\{ \begin{array}{l} \dot{x}_1 = f_1(x_1(t), x_2(t)) \\ \dot{x}_2 = f_2(x_1(t), x_2(t)) \end{array} \right\}$$

An **equilibrium point,** or **fixed point**, or **critical point,** or **rest point,** or **steady state** of the system is a point x^* such that $f(x^*) = 0$, or equivalently a point x^*: $\dot{x}(t) = 0$

The n=1 case

Consider the autonomous ODE $\dot{x} = x(1-x)$.
The differential equation gives a formula for the slope. In this example, the slope just depends on the independent variable.
The slope field gives us a rough idea about solutions to the differential equation, since solutions to the differential equation are tangent to the small slope lines.
Find equilibrium points: $\dot{x} = 0 \Rightarrow x(1-x) = 0$
Equilibrium points: $x = 0$, $x = 1$

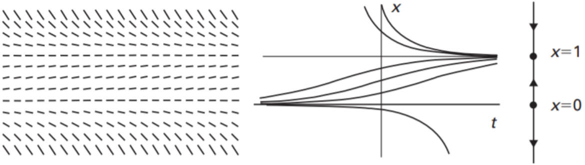


Figure: The slope field, solution graphs and phase line for $\dot{x} = x(1-x)$

$x = 1$: Stable equilibrium point (often called attractor or sink)
$x = 0$: Unstable equilibrium point (also known as repeller or source)

The n=1 case

Consider the autonomous ODE $\dot{x} = x - x^3$.
Find equilibrium points: $\dot{x} = 0 \Rightarrow x(1 - x^2) = 0$
Equilibrium points: $x = 0$, $x = 1$, $x = -1$

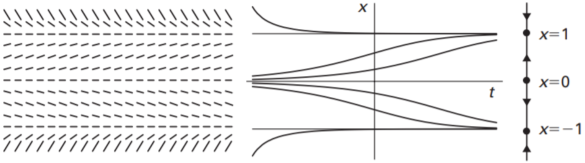


Figure: The slope field, solution graphs and phase line for $\dot{x} = x - x^3$

$x = 1$ and $x = -1$: Stable equilibrium points
$x = 0$: Unstable equilibrium point

The n=2 case

Consider the linear system with constant coefficients:

$$\dot{x} = Ax + b$$

The equilibrium point is defined as:

$$x^* : \dot{x} = 0 \ or \ x^* = -A^{-1}b$$

The equilibrium point is globally asymptotically stable if and only if the real parts of the eigenvalues (characteristic roots) of A are negative. Matrix A is then called a stable matrix.

$$\lambda_1, \ \lambda_2 = \frac{1}{2}[trA \pm \sqrt{\Delta}], \ \Delta = (trA)^2 - 4|A|$$

Classification of Equilibrium Points (n=2)

Characteristic roots	$tr(A)$, $\lvert A \rvert$, Δ	Type of Equilibrium
$\lambda_1 = \lambda_2 = \lambda > 0$	$tr(A) > 0, \lvert A \rvert > 0, \Delta = 0$	Unstable proper node
$\lambda_1 = \lambda_2 = \lambda < 0$	$tr(A) < 0, \lvert A \rvert > 0, \Delta = 0$	Stable proper node
$\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 > 0$	$tr(A) > 0, \lvert A \rvert > 0, \Delta > 0$	Unstable improper node
$\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 < 0$	$tr(A) < 0, \lvert A \rvert > 0, \Delta > 0$	Stable improper node
$\lambda_1 > 0, \lambda_2 < 0$	$\lvert A \rvert < 0$	Saddle point
λ_1, λ_2 complex positive real parts	$tr(A) > 0, \Delta < 0$	Unstable focus
λ_1, λ_2 complex negative real parts	$tr(A) < 0, \Delta < 0$	Stable focus
λ_1, λ_2 complex zero real parts	$tr(A) = 0, \Delta = 0$	Center

Phase Diagram

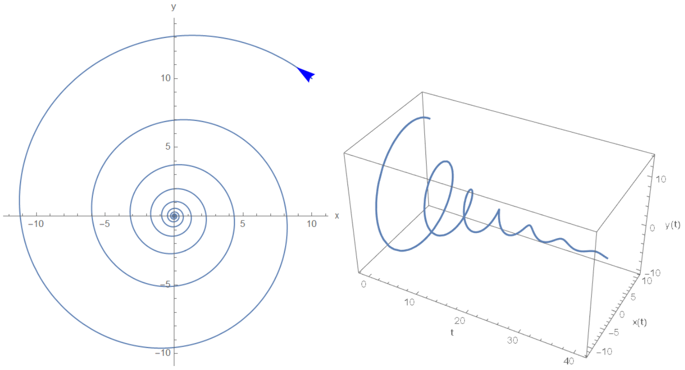


Figure: Stable focus

$$\{x' = -x - y, \quad y' = x - y, \quad x(0) = 1, \quad y(0) = 0\}$$

$$\lambda_1, \lambda_2 = -1 \pm i \qquad \Delta = -4 < 0, \qquad tr(A) < 0$$

Phase Diagram

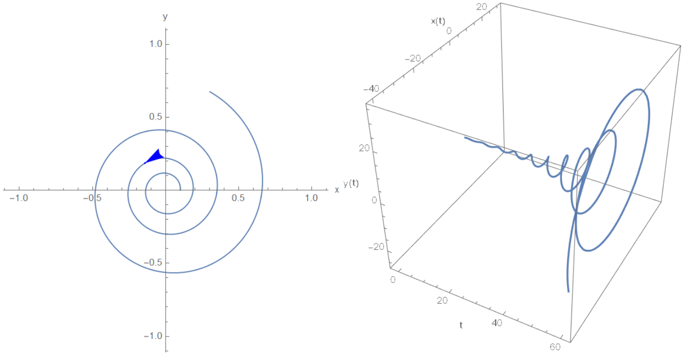


Figure: Unstable focus

$$\{x' = x - y, \quad y' = x + y, \quad x(0) = 0, \quad y(0) = 0\}$$

$$\lambda_1, \lambda_2 = 1 \pm i \qquad \Delta = -4 < 0, \qquad tr(A) > 0$$

Stable Improper Node

Consider

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Matrix A is diagonal and the system is uncoupled (since the off-diagonal elements of A are zero). Hence

$$\lambda_1 = -2 < 0, \qquad \lambda_2 = -3 < 0$$

implying

$$\lambda_1, \lambda_2 < 0 \quad and \quad \lambda_1 \neq \lambda_2$$

So, the system is stable and the steady state is a stable node i.e. orbits flow non-cyclically towards it. That is,

$$\Delta > 0, \qquad det(A) > 0, \qquad tr(A) < 0$$

Draw the phase diagram

$$\dot{x}_1(t) = 0 : \dot{x}_1(t) = -2x_1(t) + 2 = 0 \Leftrightarrow x_1(t) = 1$$

$$\dot{x}_2(t) = 0 : \dot{x}_2(t) = -3x_2(t) + 6 = 0 \Leftrightarrow x_2(t) = 2$$

Therefore $(\overline{x}_1, \overline{x}_2) = (1, 2)$ is the steady state. Moreover:

$$\frac{\partial \dot{x}_1(t)}{\partial x_1(t)} = -2 < 0$$

i.e. $\dot{x}_1(t)$ decreases as $x_1(t)$ increases (convergence). Thus, the directional arrows point $\left[\ +,\quad 0,\quad -\ \right]$ as we move $W \rightarrow E$ along the x_1 axis . Furthermore

$$\frac{\partial \dot{x}_2(t)}{\partial x_2(t)} = -3 < 0$$

i.e. $\dot{x}_2(t)$ decreases as $x_2(t)$ increases (convergence). Thus, the directional arrows point $\begin{bmatrix} - \\ 0 \\ + \end{bmatrix}$ as we move $S \rightarrow N$ along the x_2 axis.

Stable Improper Node

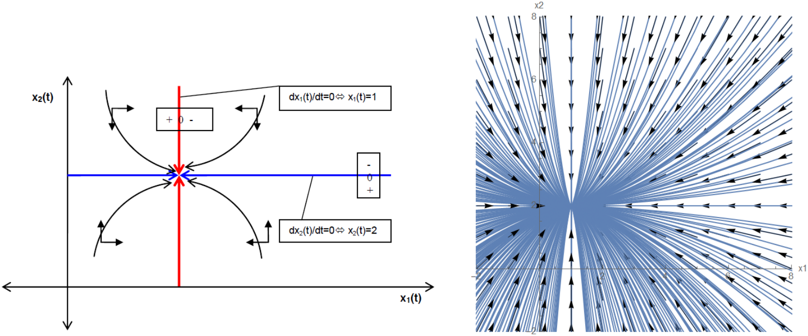


Figure: Stable Improper Node - Phase Diagram

Unstable Improper Node

Consider

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

This example is the opposite of the "stable improper node" example. The eigenvalues are:

$$\lambda_1 = 2 > 0, \qquad \lambda_2 = 3 > 0$$

Implying

$$\lambda_1, \lambda_2 > 0 \quad \textit{and} \quad \lambda_1 \neq \lambda_2$$

So, the system is unstable and the steady state is an unstable improper node i.e. orbits flow non-cyclically away from it. That is,

$$\Delta > 0, \qquad det(A) > 0, \qquad tr(A) > 0$$

Draw the phase diagram

$$\dot{x}_1(t) = 0 : \dot{x}_1(t) = 2x_1(t) - 2 = 0 \Leftrightarrow x_1(t) = 1$$

$$\dot{x}_2(t) = 0 : \dot{x}_2(t) = 3x_2(t) - 6 = 0 \Leftrightarrow x_2(t) = 2$$

Therefore $(\overline{x}_1, \overline{x}_2) = (1, 2)$ is the steady state (as in the stable node example). Moreover:

$$\frac{\partial \dot{x}_1(t)}{\partial x_1(t)} = 2 > 0$$

i.e. $\dot{x}_1(t)$ increases as $x_1(t)$ increases (divergence). Thus the directional arrows point $\left[\ -,\quad 0,\quad +\ \right]$ as we move $W \rightarrow E$ along the x_1 axis . Furthermore,

$$\frac{\partial \dot{x}_2(t)}{\partial x_2(t)} = 3 > 0$$

i.e. $\dot{x}_2(t)$ increases as $x_2(t)$ increases (divergence). Thus the directional arrows point $\begin{bmatrix} + \\ 0 \\ - \end{bmatrix}$ as we move $S \rightarrow N$ along the x_2 axis.

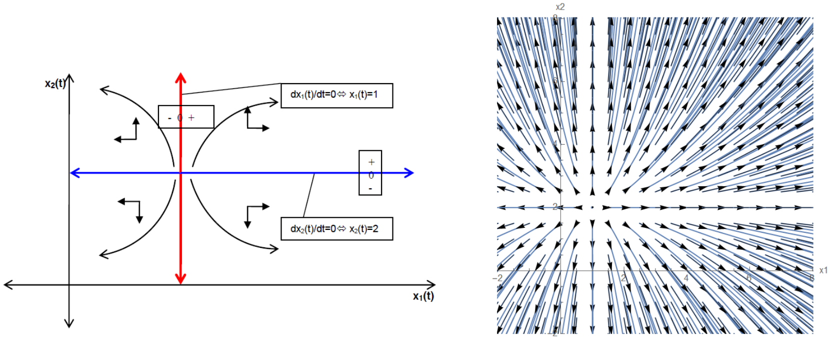


Figure: Unstable Improper Node - Phase Diagram

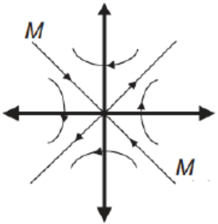
Saddle Point Equilibrium

Of special interest in economics is the **saddle point equilibrium** occurring when one of the characteristic roots is positive while the other is negative. In this case the general solution of the homogeneous system is:

$$\left\{ \begin{array}{l} x_1(t) = v_{11} c_1 e^{\lambda_1 t} + v_{21} c_2 e^{\lambda_2 t} \\ x_2(t) = v_{12} c_1 e^{\lambda_1 t} + v_{22} c_2 e^{\lambda_2 t} \end{array} \right\}$$

$$\lambda_1 \to \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} \qquad \lambda_2 \to \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} : \textit{constants}$$

In a saddle point equilibrium the system converges towards equilibrium only along the trajectory MM, which is called the stable arm of the equilibrium. The other arm is the unstable arm.



Saddle Point

Consider

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -2 \\ -\frac{1}{2} \end{bmatrix}$$

Find the characteristic polynomial:

$$|A - \lambda I| = 0 \Leftrightarrow \lambda^2 - \frac{1}{4} = 0 \Leftrightarrow (\lambda - \frac{1}{2})(\lambda + \frac{1}{2}) = 0$$

where

$$\lambda_u = 0.5 > 0 \quad and \quad \lambda_s = -0.5 < 0$$

So, the steady state is a saddle point, hence unstable. That is,

$$det(A) = -\frac{1}{4} < 0$$

Saddle Point

Find the eigenvectors

$$\begin{bmatrix} 0-\frac{1}{2} & 1 \\ \frac{1}{4} & 0-\frac{1}{2} \end{bmatrix} \begin{bmatrix} v_{u1} \\ v_{u2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{r} -\frac{1}{2}v_{u1} + v_{u2} = 0 \\ \frac{1}{4}v_{u1} - \frac{1}{2}v_{u1} = 0 \end{array}$$

So, we have that $v_{u1} = 2v_{u2}$.Therefore

$$v_u = \begin{bmatrix} v_{u1} \\ v_{u2} \end{bmatrix} = \begin{bmatrix} 2v_{u2} \\ v_{u2} \end{bmatrix} = v_{u2} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So, the eigenvector corresponding to the unstable root $\lambda_1 = \frac{1}{2}$ is

$$v_u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and the second eigenvector:

$$\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} v_{s1} \\ v_{s2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{r} \frac{1}{2}v_{s1} + v_{s2} = 0 \\ \frac{1}{4}v_{s1} + \frac{1}{2}v_{s2} = 0 \end{array}$$

So, we have that $v_{s1} = -2v_{s2}$. Therefore

$$v_s = \begin{bmatrix} v_{s1} \\ v_{s2} \end{bmatrix} = \begin{bmatrix} -2v_{s2} \\ v_{s2} \end{bmatrix} = v_{s2} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

So, the eigenvector corresponding to the stable root $\lambda_s = -\frac{1}{2}$ is

$$v_s = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Next, find the steady state (i.e. $\dot{x}_1(t) = 0$ and $\dot{x}_2(t) = 0$)

$$\bar{x}_1 = 2, \;\; \bar{x}_2 = 2$$

Write the general solution in deviation from the steady state

$$x(t) - \bar{x} = c_u v_u e^{\lambda_u t} + c_s v_s e^{\lambda_s t}$$

The general solution is: $\left\{ \begin{array}{c} x_1(t) = c_u 2e^{0.5t} - c_s 2e^{-0.5t} + 2 \\ x_2(t) = c_u e^{0.5t} + c_s e^{-0.5t} + 2 \end{array} \right\} \Leftrightarrow$

$$\left\{ \begin{array}{c} x_1(t) - 2 = c_u 2e^{0.5t} - c_s 2e^{-0.5t} \\ x_2(t) - 2 = c_u e^{0.5t} + c_s e^{-0.5t} \end{array} \right\}$$

Saddle Point

Saddle path (asymptotic) stability requires:

$$c_u = 0$$

so that

$$\lim_{t \to +\infty} (x(t) - \overline{x}) = c_u v_u e^{\lambda_u t} = 0$$

Next draw the phase diagram

Demarcation lines:

$$\dot{x}_1(t) = 0 : \dot{x}_1(t) = x_2(t) - 2 = 0 \Leftrightarrow x_2(t) = 2$$

$$\dot{x}_2(t) = 0 : \dot{x}_2(t) = 0.25x_1(t) - 0.5 = 0 \Leftrightarrow x_1(t) = 2$$

Draw directional arrows using the vector field:

$$\left[\begin{array}{c} (x_1(t), x_2(t)) = (3,1) \Rightarrow (\dot{x}_1(t), \dot{x}_2(t)) = (-1, 0.25) \text{ implying movement}(\leftarrow, \uparrow) \\ (x_1(t), x_2(t)) = (1,1) \Rightarrow (\dot{x}_1(t), \dot{x}_2(t)) = (-1, -0.25) \text{ implying movement}(\leftarrow, \downarrow) \\ (x_1(t), x_2(t)) = (3,3) \Rightarrow (\dot{x}_1(t), \dot{x}_2(t)) = (1, 0.25) \text{ implying movement}(\rightarrow, \uparrow) \\ (x_1(t), x_2(t)) = (1,3) \Rightarrow (\dot{x}_1(t), \dot{x}_2(t)) = (1, -0.25) \text{ implying movement}(\rightarrow, \downarrow) \end{array} \right]$$

Saddle Point

Streamlines/Orbits: As far as $\dot{x}_1(t) = 0$ is concerned, (asymptotic) movement is indicated by the vertical arrows. As far as $\dot{x}_2(t) = 0$ is concerned, (asymptotic) movement is indicated by the horizontal arrows.

Finally, draw the manifolds:

$$\frac{x_2(t) - \overline{x}_2}{x_1(t) - \overline{x}_1} = \frac{v_{s2}}{v_{s1}} = \frac{1}{-2} \Leftrightarrow x_2(t) = 3 - 0.5x_1(t) \; (\textit{Stable manifold})$$

and

$$\frac{x_2(t) - \overline{x}_2}{x_1(t) - \overline{x}_1} = \frac{v_{u2}}{v_{u1}} = \frac{1}{2} \Leftrightarrow x_2(t) = 1 + 0.5x_1(t) \; (\textit{Unstable manifold})$$

Saddle Point - Phase Diagram

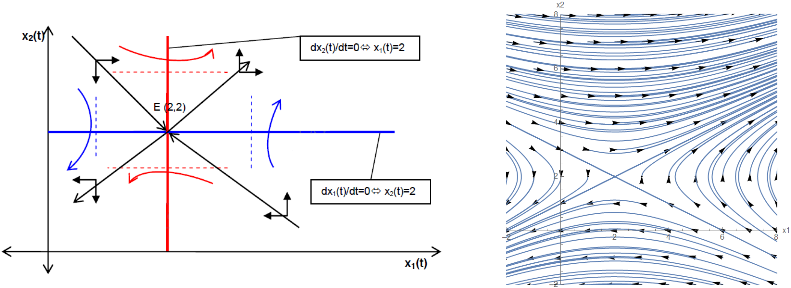


Figure: Saddle Point - Phase Diagram

Saddle Point

Consider

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

the eigenvalues are

$$\lambda_u = 2 > 0 \quad \textit{and} \quad \lambda_s = -2 < 0$$

So, the steady state is a saddle point, hence unstable. The eigenvectors are:

$$\lambda_u \rightarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \lambda_s \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

the general solution:

$$X = c_u \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t} + c_s \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

Saddle Point - Phase Diagram

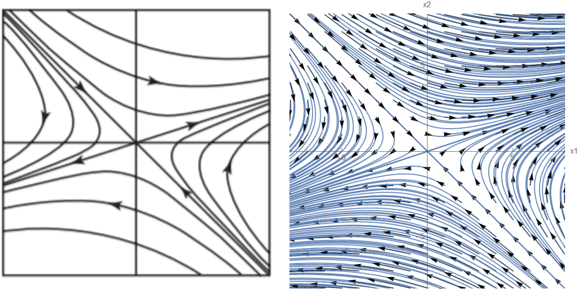


Figure: Saddle Point - Phase Diagram

Stability of nonlinear systems: Qualitative analysis (Linearization)

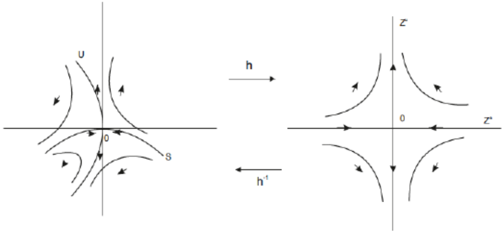
Consider the system of nonlinear differential equations $\dot{x}(t) = f(x(t)),\ f : R^n \to R^n$. Assume that x^* is an isolated equilibrium point $f(x^*) = 0$. Take the first-order Taylor expansion around the equilibrium point. The linearized system can be obtained as

$$\dot{x}(t) = f(x^*) + A(x(t) - x^*)$$

$$\dot{x}(t) = A(x - x^*),\ A = \left[\frac{\partial f(x_i^*)}{\partial x_j}\right]_{ij} = Df(x^*),\ i, j = 1, \ldots, n$$

Where A is the Jacobian matrix of the system evaluated at the equilibrium point. An equilibrium point x^* is called hyperbolic if $A = Df(x^*)$ has no eigenvalues with zero real parts. An equilibrium point x^* is called non-hyperbolic if at least one eigenvalue of $A = Df(x^*)$ has zero real part. If a hyperbolic equilibrium point is globally stable in the liner approximation, then it is locally stable at the original nonlinear system. The converse however is not necessarily true.

Stability of nonlinear systems: Qualitative analysis (Linearization)



The Ramsey optimal growth model

Solve the following optimal growth problem. All quantities are in per capita terms. Utility is concave and the production function is classical CRS.

$$\max_{\{c_t\}} \int_0^\infty e^{-\theta t} u(c(t)) dt \quad (1)$$

$$s.t. \quad \dot{k}(t) = f(k(t)) - c(t) - nk(t) \quad (2)$$

$$k(0) = k_0$$

Use the current-value formulation:

$$H^{cv}(t, c(t), k(t)) = u(c(t)) + \mu(t)(f(k(t)) - c(t) - nk(t)) \quad (3)$$

According to Maximum Principle:

$$\frac{\partial H^{cv}}{\partial c(t)} = 0 \Rightarrow u'(c(t)) = \mu(t) \Rightarrow \quad (4)$$

$$\dot{\mu}(t) = u''(c(t))\dot{c}(t) \quad (5)$$

The Ramsey optimal growth model

From the Maximum Principle Condition:

$$\dot{\mu}(t) = -\frac{\partial H^{cv}}{\partial k(t)} + \theta\mu(t) \Rightarrow \dot{\mu}(t) = -\mu(t)(f'(k(t)) - n) + \theta\mu(t) \quad (6)$$

Plugging (5) into (6) yields the Euler equation:

$$u''(c(t))\dot{c}(t) = -\mu(t)(f'(k(t)) - n) + \theta\mu(t) \overset{(4)}{\Longrightarrow} \quad (7)$$

$$u''(c(t))\dot{c}(t) = -u'(c(t))(f'(k(t)) - n - \theta) \Leftrightarrow \quad (8)$$

$$\frac{\dot{c}(t)}{c(t)} = -\frac{u'(c(t))}{u''(c(t))}\frac{1}{c(t)}(f'(k(t)) - n - \theta) \quad (9)$$

where

$$\sigma := \sigma(c(t)) = -\frac{u'(c(t)) > 0}{u''(c(t)) < 0}\frac{1}{c(t) > 0} > 0 \quad (10)$$

Hence

$$\frac{\dot{c}(t)}{c(t)} = \sigma(f'(k(t)) - n - \theta) \tag{11}$$

the Euler Equation. Moreover,

$$\dot{k}(t) = f(k(t)) - c(t) - nk(t) \tag{12}$$

the state motion, and

$$\lim_{t \to \infty} \mu(t) e^{-\theta t} k^*(t) = 0 \tag{13}$$

or using (4)

$$\lim_{t \to \infty} u'(c(t)) e^{-\theta t} k^*(t) = 0 \tag{14}$$

which implies that it would not be optimal to end up with positive capital because it could be consumed instead, since marginal utility of consumption, $u'(c(t))$, and its present value $u'(c(t))e^{-\theta t}$, is positive by assumption (concave utility function). Hence, we forced, terminal per capita capital to be zero.

The Ramsey optimal growth model

In steady state, we obtain a non-linear canonical system:

$$\left\{ \begin{array}{c} \dot{c}(t) = \sigma(c(t)) \cdot c(t) \cdot (f'(k(t)) - n - \theta) = 0 \\ \dot{k}(t) = f(k(t)) - c(t) - nk(t) = 0 \end{array} \right\} \Leftrightarrow$$

$$\left\{ \begin{array}{c} f'(k^*) = n + \theta \\ c^* = f(k^*) - nk^* \end{array} \right\}$$

where

$$f(k^*) = n + \theta \tag{15}$$

is called the Modified Golden Rule.

The phase-space, $k(t) - c(t)$, is divided in 4 regions. The stability conditions imply the following movement (directional arrows/ flow):

$$\frac{\partial \dot{k}(t)}{\partial c(t)} = -1 < 0, \qquad \textit{convergence} \qquad \textit{Horizontal flow} \tag{16}$$

i.e. as $c(t)$ increases, $\dot{k}(t)$ implies movement in the opposite direction $\left[\ +, \quad 0 \quad , -\ \right]$. Thus the horizontal directional arrows point $\rightarrow$ below $\dot{k}(t) = 0$ and $\leftarrow$ above it.

$$\frac{\partial \dot{c}(t)}{\partial k(t)} = \underset{>0}{\sigma} \cdot \underset{>0}{c(t)} \cdot \underset{<0}{f''(t)} < 0, \qquad \textit{convergence} \qquad \textit{Verical flow} \tag{17}$$

i.e. as $k(t)$ increases, $\dot{c}(t)$ implies movement in the opposite direction $\left[\ +, \quad 0, \quad -\ \right]$. Thus the vertical directional arrows point $\uparrow$ to the left and $\downarrow$ to the right of $\dot{c}(t) = 0$.

␣

Local stability

Linearise system (15) using first order Taylor's expansion around the steady state

$$\dot{c}(t) \simeq \frac{\partial \dot{c}(t)}{\partial k(t)} \mid_{k^*,c^*} [k(t) - k^*] + \frac{\partial \dot{c}(t)}{\partial c(t)} \mid_{k^*,c^*} [c(t) - c^*]$$

$$\dot{c}(t) \simeq \sigma(c^*) \cdot c^* \cdot f''(k^*)[k(t) - k^*] + 0[c(t) - c^*]$$

$$\dot{c}(t) \simeq -\beta \cdot (k(t) - k^*)$$

define $\sigma(c^*) \cdot c^* \cdot f''(k^*) := -\beta < 0,$ since $\sigma > 0$ and

$$\dot{k}(t) \simeq \frac{\partial \dot{k}(t)}{\partial k(t)} \mid_{k^*,c^*} [k(t) - k^*] + \frac{\partial \dot{k}(t)}{\partial c(t)} \mid_{k^*,c^*} [c(t) - c^*]$$

$$\dot{k}(t) \simeq [f'(k^*) - n][k(t) - k^*] - [c(t) - c^*]$$

$$\dot{k}(t) \simeq \theta(k(t) - k^*) - (c(t) - c^*)$$

Hence the linearized system

$$\begin{bmatrix} \dot{c}(t) \\ \dot{k}(t) \end{bmatrix} = \begin{bmatrix} 0 & -\beta < 0 \\ -1 & 0 < \theta < 1 \end{bmatrix} \begin{bmatrix} c(t) \\ k(t) \end{bmatrix} + \begin{bmatrix} \beta k^* \\ c^* - \theta \kappa^* \end{bmatrix}$$

whose characteristic polynomial

$$\lambda^2 - tr(J|_*)\lambda + det(J|_*) = 0$$

with

$$\Delta = [-tr(J|_*)]^2 - 4det(J|_*) = \theta^2 + 4\beta > 0$$

and

$$det(J|_*) = -\beta < 0$$

the steady state will be a saddle point.
There are two ways to find the general solution:
a) Either solve the linearized first order 2×2 system
b) Or, proceed as follows:

Local stability

Differentiate linearized $\dot{k}(t)$ once with respect to time and solve a second order linear differential equation:

$$\ddot{k}(t) = \frac{d}{dt}\dot{k}(t) = \theta\dot{k}(t) - \dot{c}(t) = \theta\dot{k}(t) - (-\beta)(k(t) - k^*)$$

$$\ddot{k}(t) - \theta\dot{k}(t) - \beta k(t) = -\beta k^*$$

The characteristic polynomial of the homogeneous equation is:

$$\rho^2 - \theta\rho - \beta = 0$$

with

$$\Delta = \theta^2 + 4\beta > 0$$

and

$$\rho_{1,2} = \frac{\theta \pm \sqrt{\theta^2 + 4\beta}}{2}$$

with two opposite-signed roots

$$\rho_2 < 0 < \rho_1$$

Local stability

The solution of the homogeneous equation is:

$$k(t) = A_1 e^{\rho_1 t} + A_2 e^{\rho_2 t}$$

which $\rho_1 > 0$ is the unstable root and $\rho_2 < 0$ is the stable root. The non-homogeneous equation has solution

$$-\beta \bar{k} = -\beta k^* \Leftrightarrow \bar{k} = k^*$$

Thus, the general solution (stated in deviation from the steady state values) equals:

$$k(t) - k^* = A_1 e^{\rho_1 t} + A_2 e^{\rho_2 t}$$

Saddle-path stability requires

$$A_1 := 0$$

Finally, use the initial condition to determine arbitrary constant A_2:

$$k(0) = k_0 \Leftrightarrow A_2 = k_0 - k^*$$

Hence, the stable solution path:

$$k(t) - k^* = (k_0 - k^*)e^{\rho_2 t}$$

Phase Diagram