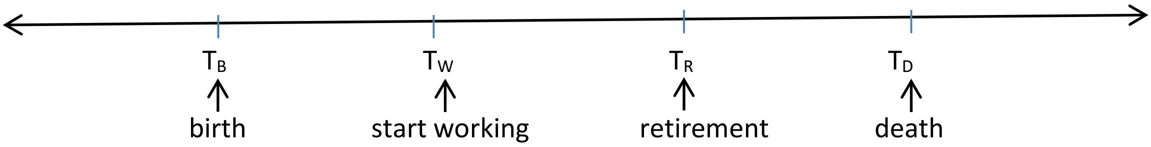
FULLY FUNDED

Timeline



$$\alpha_{t+1} = (i + r_t)\ \alpha_t\ + c_t \quad ; \quad t \in \{T_w, T_{w^{+1}}, T_{R^{-1}}\} \tag{1}$$

$$\alpha_{t+1} = (i + r_t)\ \alpha_t\ - w_t \quad ; \quad t \in \{T_R, T_{R^{+1}}, T_{D^{-1}}\} \tag{2}$$

α_t : real assets of individual or balance of the fund of the individual at the beginning of period t

r_t : real rate of return during period t

c_t : contribution to the fund in period t

w_t : withdrawals from the fund in period t

(1) $\quad \Rightarrow \quad a_{t+2} = (1 + r_{t+1})\ a_{t+1} + c_{t+1}$ (3)

(3) & (1) $\quad \Rightarrow \quad a_{t+2} = (1 + r_{t+1})\ (1 + r_t)\ a_t + (1 + r_{t+1}) c_t \ + \ c_{t+1}$ (4)

(1) $\quad \Rightarrow \quad a_{t+3} = (1 + r_{t+2})\ a_{t+2} + c_{t+2}$ (5)

(5)&(4) $\quad \Rightarrow \quad a_{t+3} = (1 + r_{t+2})(1 + r_{t+1})(1 + r_t)\ a_t + (1 + r_{t+2})(1 + r_{t+1}) c_t\ (1 + r_{t+2})\ c_{t+1} + c_{t+2}$ (6)

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$$\begin{aligned} a_{t+T} = {} & (1 + r_{t+T-1})\ \dots\ (1 + r_t)\ \alpha_t \quad + \\ & (1 + r_{t+T-1})\ \dots\ (1 + r_{t+1}) c_t \quad + \\ & (1 + r_{t+T-1})\ \dots\ (1 + r_{t+2}) c_{t+1} \quad + \\ & \dots\dots\dots + \\ & (1 + r_{t+T-1}) c_{t+T-2} \quad + \\ & c_{t+T-1} \end{aligned} \tag{7}$$

In view of (7), for $t=T_W$ and $t+T=T_R$, the accumulated real assets of the individual at the time of her retirement (I,e., the balance of her fund at retirement) are:

$$\begin{aligned} a_{T_R} = {} & (1 + r_{T_R-1}) \dots (1 + r_{T_w}) a_{T_w} \ + \\ & (1 + r_{T_R-1}) \dots (1 + r_{T_w+1}) c_{T_w} \ + \\ & (1 + r_{T_R-1}) \dots (1 + r_{T_w+2}) c_{T_w+1} \ + \\ & (1 + r_{T_R-1})\ c_{T_R-2} \ + \ c_{T_R-1} \end{aligned} \tag{8}$$

Using the product and sum operators, (8) can be re-expressed as:

$$a_{T_R} = \prod_{j=T_w}^{T_R-1}(1+r_j)\ a_{T_w} \quad + \sum_{j=T_w}^{T_R-2}\prod_{i=j+1}^{T_R-1}(1+r_i)\ c_j\ + c_{T_R-1} \tag{9}$$

Assume that the real return on the accumulated assets and the contributions made during the work period of the individual are constant. That is,

$$r_j = r \quad , \quad \forall j \in \{\ T_w, \ldots, T_{R-1}\ \} \quad , \tag{10}$$

$$c_j = c \quad , \quad "\ \forall j \in \{T_w, \ldots, T_{R-1}\ \} \tag{11}$$

In view of (10) and (11), (9) simplifies to:

$$a_{T_R} = (1+r)^{T_R-T_w} a_{T_w} + c \sum_{i=0}^{T_R-T_w-1} (1+r)^{T_R-T_w-1-i}$$

or

$$a_{T_R} = (1+r)^{T_R-T_w}\ a_{T_w} + \frac{(1+r)^{T_R-T_w} - (1+r)}{r(1+r)} c \tag{12}$$

Applying the above procedure to (2), yields:

$$a_{T_D} = (1+r)^{T_D-T_R}\ a_{T_R} - \frac{(1+r)^{T_D-T_R} - (1+r)}{r(1+r)} w \tag{13}$$

And, solving (13) for the balance of the individual's fund at retirement, gives:

$$a_{T_R} = (1+r)^{-(T_D-T_R)}\ a_{T_D} + \frac{1-(1+r)^{-(T_D-T_R-1)}}{r(1+r)} w$$

Therefore, in the fully funded system, the following condition must hold:

$$(1+r)^{T_R-T_w}\ a_{T_w} + \frac{(1+r)^{T_R-T_w} - (1+r)}{r(1+r)} c = (1+r)^{-(T_D-T_R)}\ a_{T_D} + \frac{1-(1+r)^{-(T_D-T_R)}}{r(1+r)} w \tag{14}$$

If we further assume that the individual's real assets at the start of her work period (i.e., no inheritance) and at the end of her life are zero (i.e., no bequests), then (14) requires:

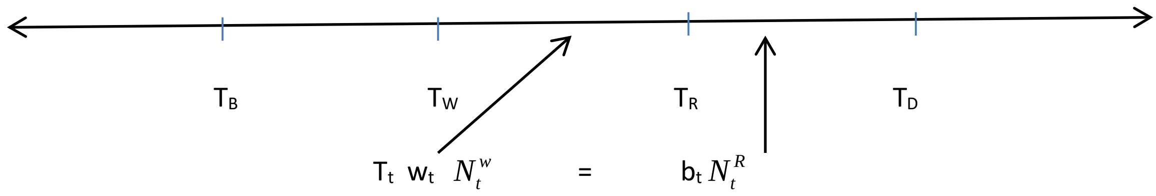
$$\frac{(1+r)^{T_R-T_w} - (1+r)}{r(1+r)} c = \frac{1-(1+r)^{-(T_D-T_R-1)}}{r(1+r)} w \tag{15}$$

That is, at the date of retirement, the discounted present value of contributions (i.e., current fund balance at the time), must be equal to the discounted present value of withdrawals. And the withdrawals to the contributions ratio, must satisfy the following:

$$\frac{w}{c} = \frac{(1+r)^{T_R-T_w} - (1+r)}{1-(1+r)^{-(T_D-T_R-1)}} \tag{16}$$

UNFUNDED

Timeline



t_t : social security

w_t : wage rate

N_t^w : number of employed workers

b_t : benefits of retired workers

N_t^R : number of retired

$$b_t = b\ w_t \tag{4}$$

$$t_t = t \tag{5}$$

(3), (5) $\Rightarrow$ $$\frac{N_t^w}{N_t^R} = \frac{b}{t} \quad \text{constant} \tag{6}$$

Demographic Problem: $\frac{N_t^w}{N_t^R} \downarrow$ so $b\ and\ or\ t$ must adjust. An increase in t diminishes labor input, capital (as the decline in labor input lowers the marginal product of capital), output and growth. A decrease in t is politically difficult (median voter retired or close to retirement) and unfair (implicit contract between old and young).

Optimal Consumption over Time and Savings

$$\max_{c_1, c_2 \geq 0} \; u\,(c_1, c_2)$$

s.t. $\quad p_1 c_1 \leq \; w - s$

$\qquad\; p_2 c_2 \; \leq (1+i)\; s$

where c_1 : consumption in period 1

c_2 : consumption in period 2

w : labor income in period 1

s : savings in period 1

i: nominal interest rate

$$s = w - p_1\, c_1$$

$$p_2 c_2 \leq (1+i)\,(w - p_1 c_1)$$

$$c_2 \leq (1+i)\left(\frac{w}{p_2} - \frac{p_1}{p_2} c_1\right)$$

$$c_2 \;\leq\; (1+i)\left(\frac{w}{p_2} - \frac{p_1}{p_2} \quad c_1\right)$$

$$\frac{p_2}{p_1} = \frac{p_2}{p_1} - \frac{p_1}{p_1} + 1$$

$$= \frac{p_2 - p_1}{p_1} + 1$$

$$= 1 + \pi$$

where : π is the inflation rate

$$c_2 \leq (1+i)\left(\frac{w}{p_1}\frac{p_1}{p_2} - \frac{p_1}{p_2} c_1\right)$$

$$c_2 \leq \frac{1+i}{1+\pi}(w - c_1)$$

$$c_2 \leq (1+r)(w - c_1)$$

where : $r \approx i - \pi$ is the real interest rate.

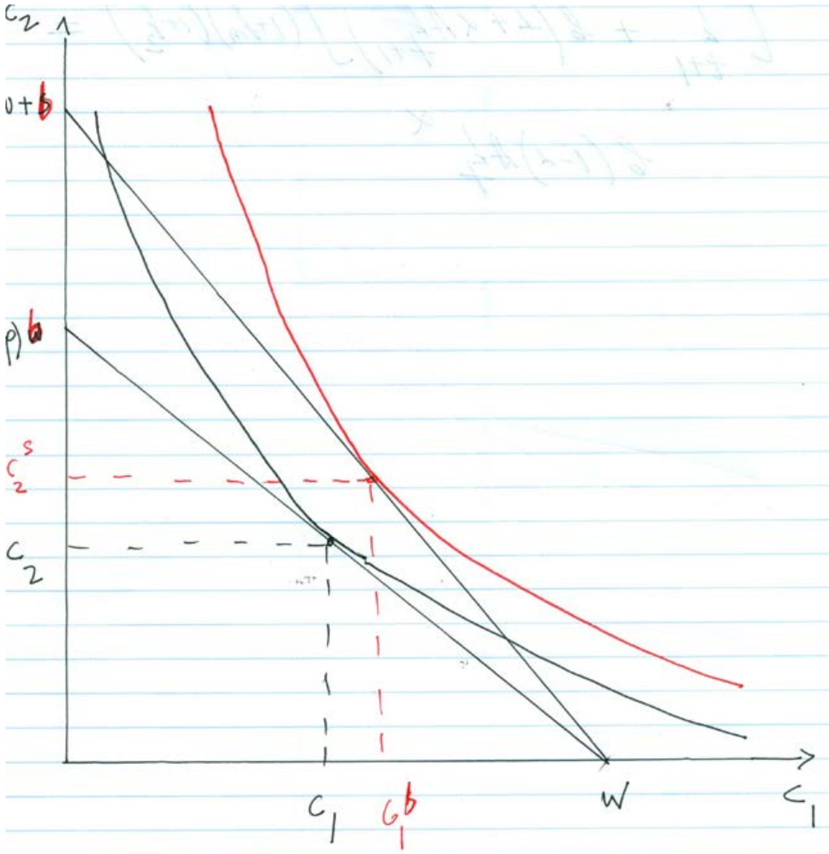
Individual's Problem:

$$\max_{c_1, c_2 \geq 0} u(c_1, c_2)$$

s.t. $c_1 + \dfrac{c_2}{(1+r)} \leq w$

Solution:

$$\frac{u_{c_1}}{u_{c_2}} = (1+r)$$



Crowding out of private Savings