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Application of Brouwer's Fixed Point Theorem: existence of Walrasian general equilibrium in an exchange economy

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We consider a closed, pure exchange (no production) economy without a public sector that is comprised of $i = 1,...,N$ economic agents. There exist $k = 1,...,L$ goods, over which the agents have preferences. Each agent has an initial exogenously given finite endowment of goods $\omega_i \in \mathbb{R}_+^L \setminus \{0_L\}$ (measured in physical units), $\omega_i = (\omega_{1i},...,\omega_{Li})$. Goods are exchanged in competitive markets through a price mechanism that translates quantities into values. Agents want to maximize their utility from the consumption of goods, given their endowment and the prices of goods.

An **equilibrium** of this economy is *defined* as a **price vector** $p = (p_1,..., p_L)$**,** $p \in \mathbb{R}_+^L$ (containing a ***single*** price for each good), and an allocation of quantities of goods over all consumers, such that

1) Each agent maximizes its utility given their endowment and the equilibrium price vector,

2) Total quantity demanded for each good equals the good's total supply (here total endowment), at the equilibrium price vector.

A more compact verbal description of this definition of equilibrium is

"At the equilibrium price vector, total quantity demanded for each good is the sum of optimally derived and feasible individual quantities demanded by each individual, and it equals the exogenous total supply/endowment".

Our only goal is to prove that **an equilibrium exists.** So we are not concerned at all with how this equilibrium could be reached.

This is not a trivial exercise/assertion.

a) We want from a price vector to be compatible with the solutions to N distinct and separate L-dimensional constrained maximization problems (which as we will see, are **not** identical) producing NL demand functions (one per good per each individual),

b) For these demand functions to be such that L sums of them (per good and over N) equal exactly the L total quantities supplied (per good), which are arbitrary and exogenously given.

We start by formalizing the assumptions describing the model.

1) Assumptions on market structure. Markets are "competitive", meaning

- There is complete, perfect and symmetric information
- There are no participation costs (like entry costs, or transactional costs like for example transportation costs)
- Participants are "price takers", they solve their utility maximization problems taking prices as given (they don't have "market power", either monopolistic or monopsonistic).

  Note: the "no participation costs" could be relaxed if we assume that such costs exist but are fixed or proportional to the endowments (possibly different for each agent). Then we could *net* them out and ignore them, by accepting that these costs do not represent *income* for some other agent, because if this was the case, it would make the model one with production also. Namely, if they exist, they represent a "dead-burden" (we note that existence of Walrasian equilibrium can be proven in a model with production also).
  Price taking behavior is critical for the model we develop, and so is the "totally equitable availability of information".

2) Assumptions on goods.

- **The goods space** $X_i \subset \mathbb{R}_+^L,\ i=1,...,N$ for each agent is assumed to be a convex and compact set.
  Note: this assumption is needed in order to obtain continuity of the individual optimal demand function, even when we consider price vectors where some of the elements are exactly zero (otherwise demand will be continuous only for strictly positive price vectors, see Mas-Colell *et al.* 1995, ch. 3 Appendix A pp. 92-94).

There is a subtler way to guarantee continuity of the demand function, see Starr (2011) pp.131-136 and the discussion therein.

From an economic point of view, convexity of the consumption set reflects unlimited divisibility of goods. While this may sound unrealistic, once we think of certain obviously imperfectly indivisible goods (like, say, a car, or even an item of clothing) in terms of *the flow of services they provide*, the potential issue becomes less severe. Convexity of the consumption set is, among other things, needed so that properties like the concavity of the utility function can even be considered.

Regarding the restriction of the consumption space to a *compact* (closed and bounded) subset of $\mathbb{R}_+^L$, closedness does not appear to contradict any observed economic behavior. Boundedness can be considered a rather innocent assumption, if we assume very-very large bounds (after all we create these economic models to study real-world economies, which are finite and bounded at least at each time instance. And even if we were to consider subjective behavior no agent would realistically picture himself consuming an "infinite" quantity from a good). Explicitly, we make the following assumption:

- For $v > 0_L$ a strictly positive conformable vector, $v + \sum_{i=1}^{N} \omega_i \in X_i \ \ \forall i$.
  With this assumption we allow for the possibility that there may exist price vectors for which the demand for some good, even by a single individual, may exceed the total available supply/endowment, i.e. it allows for the existence of strictly positive excess demand, which if it persists, violates the utility maximization problem.
  With this assumption we fully "neutralize" the boundedness assumption: nothing in this assumption as qualified above introduces any "bias" in favor of the existence of equilibrium. We will discuss how the boundedness assumption interacts with the utility maximization problem in a while.

A consumption bundle for individual *i* is denoted $x_i = (x_{1i},...,x_{Li})$. A consumption Allocation, a set that includes one consumption bundle for each agent, is denoted by $x = (x_1,...,x_N)$.

- **All goods are assumed desirable by all individuals.**
  This is an important assumption. Together with local non-satiation of preferences that will be assumed in a while, it has the consequence that the equilibrium price vector, if it exists, will be strictly positive, as we will see.

3) Assumptions on preferences. The preferences of each individual:

- are rational (complete and transitive). So they can be represented by a utility function $u_i(x_i)$.
- are continuous. So the utility function is continuous.
- are locally non-satiated and so the utility function exhibits also this property.
- are strictly convex. Since also the domain is a convex set**,** this leads to $u_i(x_i)$ being strictly quasi-concave.
  (for the above see Mas-Collel *et al.* 1995, ch. 3).

  Note. We certainly assume a degree of "similar structure" in preferences, but we certainly do NOT assume that the corresponding utility functions are identical, or that they will have the same maximizer for the same endowment and price vectors. Note we do *not* assume that utility functions are differentiable.

4) Behavior. Each agent solves $\max_{x_i} u_i(x_i) \quad \text{s.t. } p\cdot x_i \le p\cdot\omega_i, \quad x_i \in X_i$

...where here the dot represents the inner product of two vectors. The above reflects the "price-taking" behavior, and also implies one more important aspect of the model:

- That *all endowments are offered to the corresponding markets as quantities supplied*, even though goods are desirable and we expect that at least some agents would want at the optimum to consume a good that the agents also possess as part of their endowment. This in principle raises the question "then why bringing it to the market, why not consuming it directly?"

The reasoning here is that agents understand that their endowments have not only utility-enhancing value, but *exchange value* also, which will determine their consumption opportunities. So they participate in the market with their full endowment in order to maximize these opportunities. Strictly speaking, this reasoning breaks down if we want to consider a price that is exactly zero, since then the good in question will not have any exchange-value. But this won't create any problems in our model as we will see in a while.

On the other hand, if we have opted for a *supply* function, i.e. a budget constraint like $p \cdot x_i \leq p \cdot s_i(p),\ \ s_i(p) \leq \omega_i$, we would lose an important property of the demand function (again, see in a while).

Then the general equilibrium is described as follows:

A price vector p^* and a consumption Allocation x^* such that

- $x_i^* = x_i\left(p^*\right)$ is the solution to $\max u_i(x_i)\ \ s.t.\ \ p^* \cdot x_i \leq p^* \cdot \omega_i,\ \ x_i \in X_i\ \ i = 1,...,N$

- $\sum_{i=1}^{N} x_{ki}\left(p^*\right) = \sum_{i=1}^{N} \omega_{ki} \quad k = 1,...,L$ (all markets "clear")

There is nothing special about markets, we do not "wish" them to clear for "their own benefit". We only care about the utility of the agents. But this condition guarantees that the optimal demands will be actually realized and consumed, realizing thus the utility maximization solutions.

A) The utility maximization problem: the Walrasian demand function.

Let $x_i(p) = \arg\max\{u_i(x): x_i \in X_i,\ p\cdot x_i \leq p\cdot\omega_i\}$, the vector-valued function of demands for each good from agent *i* that solves its utility maximization problem, given the price vector and the endowment. Note that we treat it here as depending on the price vector only and not on the endowment also, since we will not be conducting comparative statics related to changes in endowments (we treat them as fixed parameters). This is the Walrasian demand correspondence (in the context of partial equilibrium analysis, it is usually called the Marshallian demand correspondence).

Note also that, due to the boundedness assumption on X_i, we have an additional constraint to satisfy: it is conceivable that for some price vector, an individual will have optimal demand at the boundary of X_i. This certainly does not imply that the utility function, as a function, is bounded. It does imply that *attained utility* will be finite, but this is the case anyway, due to the existence of the budget constraint. We just impose an additional exogenous constraint on the individual (and not an unrealistic one, given that the bounds can be set to enormous levels and literarily outside human experience). Moreover, if demand is at the boundary, it is certain that it will exceed supply for some price vector *p:* but we don't care, as long as we can find an equilibrium price vector which by construction does not correspond to such a situation.

Important properties of $x_i(p)$ are:

- $x_i(p)$ **is homogeneous of degree zero in its argument**. This is immediate from the fact that
  $\{x_i \in X_i,\ ap\cdot x_i \leq ap\cdot\omega_i\} = \{x_i \in X_i,\ p\cdot x_i \leq p\cdot\omega_i\}\quad \forall a > 0$
  Namely, multiplying the price vector by any scalar, leaves the feasible set unchanged, and so the maximizer will be the same. Note that this property *has nothing to do with the utility function* (except perhaps for the universal assumption that prices do not enter the utility function). It reflects that in our model, only relative/normalized prices matter. This is the property that we would lose if we had attempted to use a supply function in place of the endowment.

- $x_i(p)$ **satisfies Walras' law:** $p \cdot x_i(p) = p \cdot \omega_i$.

  This is a consequence of local non-satiation: if $p \cdot x_i(p) < p \cdot \omega_i$ there exists a bundle that is feasible to purchase and is preferred over $x_i(p)$. But this contradicts $x_i(p)$ being the maximizer.

- Since $u_i(x)$ is strictly quasi-concave, $x_i(p)$ is a vector-valued ***function*****:** the solution vector, and so the optimal quantity demanded per good, will be unique per individual.

  Proof: Ad absurdum, assume that $x_i(p)$ is a correspondence and includes say two bundles, $x_i'(p) \neq x_i''(p)$. Then we must have that $u_i(x_i'(p)) = u_i(x_i''(p)) = \bar{u}$ (since they are both admissible solutions to the maximization problem). Consider the bundle $x_i'''(p) = ax_i'(p) + (1-a)x_i''(p)$. It is feasible because

$$p \cdot x_i'''(p) = p\left[ax_i'(p) + (1-a)x_i''(p)\right] = ap \cdot x_i'(p) + (1-a)px_i''(p)$$

$$= ap \cdot \omega_i + (1-a)p \cdot \omega_i = p \cdot \omega_i$$

  ...where we have also used the fact that solutions satisfy Walras' law.

  Now, **strict** quasi-concavity means, for any value and so also for $\bar{u}$ that

$$\left\{x_i'(p) \neq x_i''(p), u\left(x_i'(p)\right) \geq \bar{u}, \ u\left(x_i''(p)\right) \geq \bar{u}\right\} \Rightarrow u\left(ax_i'(p) + (1-a)x_i''(p)\right) > \bar{u}, \ \ a \in (0,1)$$

  But then we have $u(x''') > \bar{u} = u(x') = u(x'')$, which contradicts the assumption that x' and x'' belong to $x_i(p)$ (if $u(x''')$ offers higher utility and is feasible it should be a member of the correspondence $x_i(p)$ while $x_i'(p)$ and $x_i''(p)$ should not) . Therefore we conclude that under strict quasi-concavity, $x_i(p)$ consists of a single vector, and so it is a function.

- $x_i(p)$ **is continuous** by the Theorem of the Maximum (read about it). The compactness assumption we made on the consumption space, is needed to apply the Theorem of the Maximum here.

B) The Walrasian excess demand function and the condition for equilibrium.

The "excess demand function" of agent *i* is defined as

$$z_i(p) := x_i(p) - \omega_i$$

It express agent's i *net* demand for each good (positive or negative), above or below the quantities that the agent possesses in its endowment vector. The aggregate excess demand function is

$$z(p) = \sum_{i=1}^{N} z_i(p) = \sum_{i=1}^{N} \left(x_i(p) - \omega_i \right)$$

Important properties:

- $z(p)$ **is homogeneous of degree zero**, since each $x_i(p)$ has this property: $z(ap) = z(p), \quad a > 0$
- $z(p)$ **is continuous** because each $x_i(p)$ is so, and sums of continuous functions (together with shifts represented by the endowments) are themselves continuous.
- $p \cdot z(p) = 0 \;\; \forall p$ **(Walras' law).** Immediate from $p \cdot z(p) = \sum_{i=1}^{N} p\left(x_i(p) - \omega_i \right) = \sum_{i=1}^{N} \left(p \cdot x_i(p) - p \cdot \omega_i \right) = 0 + 0 + ...0 + 0 = 0$

PROPOSITION. p^* **is an equilibrium price vector for the above economy, if and only if** $z\left(p^*\right) = 0_L$ **.**

That this condition incorporates the two expressions that define the equilibrium, and so it is equivalent with them should be obvious: first it is expressed using the Walrasian demand functions, which solve the maximization problem of the agents so it already incorporates the first aspect of equilibrium. Second, it states that each single element of the vector $z\left(p^*\right)$ should be equal to the zero vector. But each element of this vector represents the excess demand at market-level for each good. So it requires that excess demand in each market is zero, which is the second aspect of the Walrasian equilibrium as defined here (which validates that the utility maximization solutions are indeed realized).

C) Existence of Walrasian Equilibrium.

To prove existence of the equilibrium, we need to prove that there exists a price vector p^* such that $z\left(p^*\right)=0_L$.

We start by exploiting the fact that the aggregate excess demand is homogeneous of degree zero in prices, $z(p)=z(ap),\ a>0$. We could then set $a=\left(\sum_{k=1}^{L} p_k\right)^{-1}$ which would lead to the elements of the transformed price vector summing up to unity.

We define the unit simplex in $\mathbb{R}_+^L$, $\Delta=\left\{p\in\mathbb{R}_+^L : \sum_{k=1}^{L} p_k=1\right\}$ and due to homogeneity of degree zero, we can restrict our search for an equilibrium price vector among price vectors in this set (note: we use the same simple *p*-notation for the transformed prices to avoid clutter).

Note. We do not do this for convenience. Δ is a convex and compact subset of $\mathbb{R}_+^L$ (while $\mathbb{R}_+^L$ is not compact). **So the ability to consider normalized prices is crucial in order to use the Brouwer FPT.** And we were able to consider Δ only because the aggregate excess demand is homogeneous of degree zero in prices, a property which in turn depends on the fact that the individual demand functions are also homogeneous of degree zero in prices, which in turn, depends on the assumption that all endowments are brought to the market, and on the convexity of the budget set $\left\{x_i\in X_i,\ p\cdot x_i\le p\cdot\omega_i\right\}$ which in turn depends on the convexity of the consumption set. This is a second important service that the convexity assumption provides.

The steps to prove that an equilibrium price vector exists are the following:

- We construct a certain function of the price vector which is a self-map in Δ.
- We verify that this function satisfies the assumptions of Brouwer's Fixed Point Theorem and conclude that it has a fixed point in Δ.
- We prove that this fixed point must be a strictly positive price vector, given the assumptions on goods and on preferences.
- We then show that this strictly positive fixed point satisfies the conditions to be a Walrasian equilibrium price vector, as defined.

Note. Contemplating the above, it should be clear that the critical step is the finding and the construction of the function, so that it both satisfies the conditions of the Brouwer Fixed Point Theorem *and* its fixed point satisfies the equilibrium condition. Guiding principles to construct such a function and related intuition are discussed later.

Constructing the function. Let $z_k(p) := \sum_{i=1}^{N} x_{ki}(p) - \sum_{i=1}^{N} \omega_{ki},\ k = 1,...,L$ be the excess demand function in the market for good *k*. $z_k(p)$ is also a continuous function. So is the function $z_k^+(p) := \max\{z_k(p), 0\}$ (**note** that here we do not maximize over *p*, we only compare $z_k(p)$ with zero and select the larger of the two). Denote by $z^+(p)$ the vector-valued function that contains as elements all $z_k^+(p),\ k = 1,...,L$. Consider now the vector-valued function

$$f(p) = \frac{p + z^+(p)}{\sum_{k=1}^{L}\left(p_k + z_k^+(p)\right)} = \frac{p + z^+(p)}{1 + \sum_{k=1}^{L} z_k^+(p)}$$

... the second equality because normalized prices satisfy $\sum_{k=1}^{L} p_k = 1$. Note that the numerator is an *L*-dimensional vector, whose elements are each divided by the scalar in the denominator. The division operation is permissible, since obviously, the denominator will never be zero. The only reason that the denominator appears is to make $f(p) = \left(f_1(p),..., f_L(p)\right)$ a self-map, i.e. to return vectors whose sum of their elements satisfy $\sum_{k=1}^{L} f_k\left(p\right) = 1$. Apart from that, the denominator plays no role in the proof of existence of equilibrium.

The denominator sums the elements of the vector appearing in the numerator, so

$$\sum_{k=1}^{L} f_k\left(p\right) = \frac{p_1 + z_1^+(p)}{\sum_{k=1}^{L}\left(p_k + z_k^+(p)\right)} + \frac{p_2 + z_2^+(p)}{\sum_{k=1}^{L}\left(p_k + z_k^+(p)\right)} + ... + \frac{p_L + z_L^+(p)}{\sum_{k=1}^{L}\left(p_k + z_k^+(p)\right)} = 1$$

So $f(p)$ is a self-map in Δ, which is a convex and compact subset of $\mathbb{R}_+^L$. Moreover, $f(p)$ is a continuous function given the continuity of its components and how they are combined.

Brouwer Fixed Point Theorem. We see that the function $f(p)$ satisfies the conditions of Brouwer's FPT. So it has a fixed point $p^* \in \Delta : f\left(p^*\right) = p^*$.

Note. The application of Brouwer's FPT ends here. But it offers an **indispensable** service, by asserting the existence of a fixed point. This permits us to substitute p^* for $f\left(p^*\right)$, and we will use this equality in order to prove the existence of equilibrium.

The fixed point is a strictly positive price vector. Assume that p^* contains a single zero element, say $p_L^* = 0$. Since p^* is the fixed-point vector**,** the corresponding element of $f\left(p^*\right)$ must then satisfy

$$f_L\left(p^*\right) = \frac{p_L^* + z_L^+\left(p^*\right)}{1 + \sum_{k=1}^{L} z_k^+\left(p^*\right)} = \frac{0 + \max\left\{z_L\left(p^*\right), 0\right\}}{1 + \sum_{k=1}^{L} z_k^+\left(p^*\right)} = p_L^* = 0$$

$$\Rightarrow \max\left\{z_L\left(p^*\right), 0\right\} = 0 \;\Rightarrow\; z_L\left(p^*\right) \leq 0 \;\;\Rightarrow \sum_{i=1}^{N}\left(x_{Li}\left(p^*\right) - \omega_{Li}\right) \leq 0$$

$$\Rightarrow \sum_{i=1}^{N} x_{Li}\left(p^*\right) \leq \sum_{i=1}^{N} \omega_{Li}$$

In words, to have a zero price in the fixed-point price vector, quantity demanded of the associated good should be lower or equal than the endowment/supply at the zero price. But this contradicts the assumptions we have made on goods and preferences: all goods are desirable, and so is good *L*. At zero price, consumers, due to local non-satiation, would require the maximum possible amount of the good given their consumption set,

$$p_L^* = 0 \Rightarrow \sum_{i=1}^{N} x_{iL}\left(p^*\right) = \sum_{i=1}^{N} \max x_{Li} > \sum_{i=1}^{N} \omega_{Li}$$

...the last inequality due to the assumption we made on the consumption sets.
(**note:** a zero price raises the issue of whether any quantity of the good will be supplied to the market, but we do not need to examine this issue).

So a price vector with even a single zero element in it cannot satisfy the conditions required to be a fixed point of $f(p)$. Therefore, we conclude that the fixed point vector can only be strictly positive.

The fixed point is a Walrasian equilibrium price vector. Since the aggregate demand function satisfies Walras's law for every price vector, it satisfies it also for p^*. So we know (and this is the second time we use the fixed point property $f\left(p^*\right)=p^*$), that it holds

$$p^*\cdot z\left(p^*\right)=0=f\left(p^*\right)\cdot z\left(p^*\right)\Rightarrow\frac{p^*+z^+\left(p^*\right)}{1+\sum_{k=1}^{L}z_k^+\left(p^*\right)}\cdot z\left(p^*\right)=0$$

$$\Rightarrow p^*\cdot z\left(p^*\right)\;+\;z^+\left(p^*\right)\cdot z\left(p^*\right)=0\;\Rightarrow z^+\left(p^*\right)\cdot z\left(p^*\right)=0$$

...the first component being zero by Walras' law. The remaining term is an inner product of two vectors, and it is decomposed in

$$z_1^+\left(p^*\right)\cdot z_1\left(p^*\right)\;+\;z_2^+\left(p^*\right)\cdot z_2\left(p^*\right)+...+z_L^+\left(p^*\right)\cdot z_L\left(p^*\right)=0\,.$$

Consider the typical element of the above sum-product:

$$z_k^+\left(p^*\right)\cdot z_k\left(p^*\right)=\max\left\{z_k\left(p^*\right),0\right\}\cdot z_k\left(p^*\right).$$

Assume that $z_k\left(p^*\right)>0$. Then the typical element will be positive.

Assume that $z_k\left(p^*\right)=0$. Then the typical element will be zero.

Assume that $z_k\left(p^*\right)<0$. Then we will have $\max\left\{z_k\left(p^*\right),0\right\}=0$ and the typical element will again be zero.

In no case do we obtain that the typical element will be negative. It follows that if, even one of these products is positive, the whole product sum $z^+\left(p^*\right)\cdot z\left(p^*\right)$ cannot be zero. So we conclude that in all markets,

$$\forall k=1,...,L\quad z_k\left(p^*\right)\leq 0\Rightarrow z\left(p^*\right)\leq 0_L$$

In a more general setting, this would already prove that p^* is a Walrasian equilibrium price vector. But in our variant we want $z(p^*)=0_L$. For this we need to use the result that p^* is strictly positive.

Assume that we have $z_k(p^*)<0$ namely that in the market for good k we have excess supply for this price vector. The aggregate excess demand function satisfies Walras' law for every price vector, so we have

$$p^* \cdot z(p^*)=0 \Rightarrow p_1^* \cdot z_1(p^*) + p_2^* \cdot z_2(p^*)+...+p_L^* \cdot z_L(p^*)=0$$

If, even one $z_k(p^*)<0$, then since $p^*>0_L \Rightarrow p_k^*>0$, we will have a strictly negative element in $p^* \cdot z(p^*)$. It follows that then, for $p^* \cdot z(p^*)$ to be able to equal zero, we must have for some other good, say $m, z_m(p^*)>0$, i.e. strictly positive excess demand at a non-zero price $p_m^*>0$. But we have already obtained $z(p^*)\le 0_L \Rightarrow z_k(p^*)\le 0 \quad \forall k=1,...,L$. So assuming $z_k(p^*)<0$ for even one good leads to a contradiction, and so we conclude that $z_k(p^*)=0, \quad k=1,...,L$ which implies that $z(p^*)=0_L$.

But then the price vector p^*, except of being a fixed point for function $f(p)$, is a Walrasian equilibrium price vector also. And thus, we have proven that a Walrasian equilibrium price vector exists for this economy.

Meta-notes

A more general detailed exposition for the existence of Walrasian equilibrium can be found in Mas-Colell *et al.* (1995), ch. 17.A, 17.B, 17.C. In there the authors allow for the possibility of some prices being zero in equilibrium (so they allow for the existence of goods which may be "bads" which are then "costlessly disposed" in equilibrium), and they also treat the case of the demand being a correspondence and not necessarily a function (allowing for weaker assumptions on preferences). As a consequence, they use the Kakutani Fixed Point Theorem for the proof. They then present summarily the framework into which Brouwer's FPT is applicable. Moreover they attempt to provide some intuition as to how the function $f(p)$ could pop up in the mind of someone who is trying to prove the existence of a Walrasian equilibrium price vector.

More directly spelled intuition on the construction of the $f(p)$ function can be found in Starr (2011) ch. 5, where it becomes clear that it is a "price adjustment towards equilibrium" device, (although this is just notional: as we have said we make no assumptions whatsoever about how, or even if, the equilibrium may be reached).

Another derivation that uses Brouwer's Fixed Point Theorem (a much more brief one, that also appears to include some rather confusing typographical mistakes), can be found in Corbae *et al.* 2008 "An Introduction to Mathematical Analysis for Economic Theory and Econometrics", pp.161-163. The different assumption there is that they assume monotonic preferences and so a monotonic utility function, and they use this property in their proof.

Note: in different expositions the $f(p)$ function can be seen to differ as regards its exact functional form (or be a correspondence and not just a function), but the rationale is the same and it accomplishes the same purpose.

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