Athens University of Economics and Business
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A useful pseudometric: application to Consumer Theory.

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Let $X = \mathbb{R}_+ \times \mathbb{R}_+$ represent a space of combinations of two goods indexed by 1 and 2, respectively. Two typical elements of X are $x = (x_1, x_2)'$, $y = (y_1, y_2)'$ representing consumption bundles.

Define the following function on X,

$$d_\ell(x,y) = \begin{cases} 0 & x_1 = y_1 \\ 1 & x_1 \neq y_1 \end{cases}$$

Namely we define something that it looks like the discrete metric, but examines only one of the components of the two-dimensional vectors that are the elements of X. We examine whether $d_\ell(x,y)$ is a metric.

(i) Positivity: by its definition $d_\ell(x,y)$ is non-negative.

(ii) Separation: we see that we may have $x_1 = y_1,\ x_2 \neq y_2 \Rightarrow x \neq y$ while $d_\ell(x,y) = 0$. So this property is *not* satisfied.

(iii) Symmetry. It holds, by the symmetry of the "=" operation.

(iv) Triangle Inequality. We want to examine whether

$$d_\ell(x,y) \le d_\ell(x,z) + d_\ell(z,y), \quad x,y,z \in X$$

For the various cases we have

$$\begin{cases} x_1 = z_1 = y_1 & d_\ell(x,y) = 0 \leq 0 = d_\ell(x,z) + d_\ell(z,y) \\ x_1 = z_1 \neq y_1 & d_\ell(x,y) = 1 \leq 0 + 1 = 1 = d_\ell(x,z) + d_\ell(z,y) \\ x_1 \neq z_1 = y_1 & d_\ell(x,y) = 1 \leq 1 + 0 = 1 = d_\ell(x,z) + d_\ell(z,y) \\ x_1 = y_1 \neq z_1 & d_\ell(x,y) = 0 \leq 1 + 1 = 2 = d_\ell(x,z) + d_\ell(z,y) \\ x_1 \neq y_1 \neq z_1 & d_\ell(x,y) = 1 \leq 1 + 1 = 2 = d_\ell(x,z) + d_\ell(z,y) \end{cases}$$

So the Triangle Inequality holds.

Also, we have that $d_\ell(x,x) = 0 \quad \forall x \in \mathbb{R}_+^2$.

We conclude that $d_\ell(x,y)$ is a pseudo-metric. Is it of any use?

Consider a consumer that has *lexicographic preferences* over the good 1. This means that to order two bundles $x = (x_1, x_2)'$, $y = (y_1, y_2)'$, the consumer looks first solely on the quantities of good 1. If $x_1 > y_1$ the consumer will prefer bundle x to bundle y *irrespective* of the amounts x_2, y_2 of good 2 in the two bundles. Analogously for preference towards the other direction. If $x_1 = y_1$, then, and only then, does the consumer compares x_2, y_2 to order the bundles in terms of preference.

Lexicographic preferences may not be the most commonly observed preference structure, but they do exist and may be important for policy: consider a person that has a chronic incurable illness that is also characterized by life-threatening crises. Immediate access, quantity and quality of medical services and support are so much more important to this individual than many other goods (that he enjoys consuming also), that we could use the concept of lexicographic preferences on him.

Then our pseudo-metric $d_\ell(x,y)$ **provides us with the following useful information:**

For pairs of bundles such that $d_\ell(x,y) = 0$ we learn that the ordering of the bundles in terms of consumer preference will be dictated solely by the amount of good 2 in x and in y.

For pairs of bundles such that $d_\ell(x,y) = 1$ we learn that the ordering of the bundles in terms of consumer preference will be dictated solely by the amount of good 1 in x and in y.

Our pseudometric may not satisfy the Separation property, but it usefully "separates" the goods space! (given what we have to model, i.e. lexicographic preferences).

Generalization

One can see that, more generally, in many cases we can obtain a pseudo metric in multidimensional spaces by taking a known metric and "restrict" it so that it uses only a subset of the dimensions of the associated Set (*without restricting the set itself).*

For example, consider the Euclidean metric on $\mathbb{R}^k$ $d_I(x,y)=\sqrt{\sum_{i=1}^{k}(x_i-y_i)^2}$

and the function

$$d_{I,k-1}(x,y)=\sqrt{\sum_{i=1}^{k-1}(x_i-y_i)^2}$$

$d_{I,k-1}(x,y)$ is a pseudo-metric on $\mathbb{R}^k$ (and it obviously becomes a metric proper again if we consider $\mathbb{R}^{k-1}$).

The question is, if we ignore the *k*-th dimension of the set in the metric, why keep it in the set in the first place? The previous Consumer Theory application provides exactly an example of why one would want to have this combination, motivating the use of this kind of pseudo-metrics.

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