Athens University of Economics and Business
Department of Economics

Postgraduate Program - MSc in Economic Theory
Course: Mathematical Economics (Mathematics II)
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Problem Set 2

Open and Closed Balls, Boundness, and Total Boundness

Exercise 1

Show that open and closed balls can be defined for pseudo-metric spaces.

Exercise 2

For the following (X, d) pairs, show that they constitute (pseudo-)metric spaces and define the unit open balls on them and visualize them:

1. $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}$, such that

$$d(x, y) = \begin{cases} 0, & x = y \\ c, & x \neq y \end{cases}, \forall x, y \in X$$

   with $c > 0$.

2. $X = \mathbb{R}^2$ and $d : X \times X \to \mathbb{R}$, such that

$$d(x, y) = \sqrt{(x - y)' A(x - y)}, \forall x, y \in X$$

   with $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

3. $X = \mathbb{R}^3$ and $d : X \times X \to \mathbb{R}$, such that

$$d(x, y) = \max\left\{|x_1 - y_1|, \sqrt{(x_2 - y_2)^2 + (x_3 - y_3)^2}\right\}, \forall x, y \in X$$

*Please report any typos, mistakes, or even suggestions at zaverdasd@aueb.gr.

Exercise 3

Let (X, d) be a metric space and for some $x, y \in X$ and $\varepsilon > 0$ let $y \in \mathcal{O}_d(x, \varepsilon)$. Show that $\exists\, \delta > 0 : \mathcal{O}_d(y, \delta) \subseteq \mathcal{O}_d(x, \varepsilon)$.

Exercise 4

Let (X, d) be a metric space and $Y \subseteq X$. Define $d' : Y \times Y \to \mathbb{R}$ such that $d' = d|_{Y \times Y}$. Then (Y, d') is a metric subspace of (X, d). Show that $\mathcal{O}_{d'}(x, \varepsilon) = \mathcal{O}_d(x, \varepsilon) \cap Y$ and $\mathcal{O}_{d'}[x, \varepsilon] = \mathcal{O}_d[x, \varepsilon] \cap Y$.

Exercise 5

Is $(0, 1)$ a bounded set?

Exercise 6

Let $d : \mathbb{R}_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ such that $d(x, y) = |ln(x) - ln(y)|$, $\forall x, y \in \mathbb{R}_{++}$ be a metric on $\mathbb{R}_{++}$. Is $(0, 1)$ a d-bounded subset of $\mathbb{R}_{++}$?

Exercise 7

Let (Y, d) be a metric space and $X \neq \varnothing$. For $d_{sup} : \mathcal{B}(X, Y) \times \mathcal{B}(X, Y) \to \mathbb{R}$ such that

$$d_{sup}(f, g) = \sup_{x \in X} d(f(x), g(x)),\ \forall f, g \in \mathcal{B}(X, Y)$$

show that:

1. $(\mathcal{B}(X, Y), d_{sup})$ is a metric space.
2. If (Y, d) is bounded, then $(\mathcal{B}(X, Y), d_{sup})$ is also bounded.

Exercise 8

Let $X \subseteq \mathbb{R}^N$ with $N \in \mathbb{N}^*$ and $d : X \times X \to \mathbb{R}$ such that $d(x, y) = \left(\sum_{i=1}^{N} |x_i - y_i|^2\right)^{\frac{1}{2}}$, $\forall x, y \in X$ be the Euclidean metric on X. Show that d-boundeness in X is sufficient for d-total boundeness in X.

(Hint: Consider a d-bounded set in X and show that the ball that covers it is d-totally bounded.)

Exercise 9

Let $X = \{(x_n)_{n \in \mathbb{N}^*} : x_n \in \mathbb{R},\ \sum_{i=1}^{\infty} x_i^2 < +\infty\}$ (i.e. X is the set of square summable real sequences) and $d : X \times X \to \mathbb{R}$ such that $d(x, y) = (\sum_{i=1}^{\infty} |x_i - y_i|^2)^{\frac{1}{2}}$, $\forall x, y \in X$ be a metric on X. Show that d-boundeness in X is not sufficient for d-total boundeness in X.

(Hint: Consider the sequence $\mathbf{0} = \{0\}_{n \in \mathbb{N}^*} \in X$ and the d-closed unit ball centered at it. Use Riesz's Lemma and the Pigeonhole Principle.)

Exercise 10

Let $X = \{f : [0,1] \to \mathbb{R} | \int_0^1 f^2(x)dx < +\infty\}$ be the set of square integrable functions from $[0,1]$ to $\mathbb{R}$. Consider the metric function $d(f,g) := \left(\int_0^1 (f(x) - g(x))^2 dx\right)^{\frac{1}{2}}$ on X. If $\mathbf{0} : [0,1] \to \mathbb{R}$ is a function in X such that $\mathbf{0}(x) := 0 \ \forall x \in [0,1]$, consider $\mathcal{O}_d[\mathbf{0}, 1]$ and show that it is not d-totally bounded.

Exercise 11

Let (X_i, d_i) be metric spaces $\forall i \in \mathcal{I}$ with $\mathcal{I}$ a finite index set. For the cartesian product $X := \prod_{i\in\mathcal{I}} X_i$ there can be defined the following structured sets (X, d_Π) with $d_\Pi \in \{d_{\Pi_{max}}, d_{\Pi_I}, d_{\Pi_{||}}\}$ and d_Π are defined as

$$\begin{aligned} d_{\Pi_{max}} &= \max_{i\in\mathcal{I}} d_i \\ d_{\Pi_I} &= \left(\sum_{i\in\mathcal{I}} d_i^2\right)^{\frac{1}{2}} \\ d_{\Pi_{||}} &= \sum_{i\in\mathcal{I}} d_i \end{aligned}$$

and are appropriate metric functions on X. Let $A_i \subseteq X_i, \forall i \in \mathcal{I}$ and $A := \prod_{i\in\mathcal{I}} A_i$, which implies that $A \subseteq X$. Show that A is a d_Π-totally bounded subset of X iff A_i are d_i-totally bounded subsets of X_i $\forall i \in \mathcal{I}$, for each of the three d_Π defined above.

Exercise 12

Let d_1 and d_2 be both metrics on a non empty set X such that $d_1 \leq cd_2$ with $c > 0$. Show that:

1. for $x \in X$ and $0 < \varepsilon$ then $\mathcal{O}_{d_2}(x, \varepsilon) \subseteq \mathcal{O}_{d_1}(x, c \cdot \varepsilon)$.

2. if $A \subseteq X$ is d_2-bounded, then it is d_1-bounded.

3. if there exists $c' > 0$ such that $c'd_2 \leq d_1 \leq cd_2$, then $A \subseteq X$ is d_1-bounded iff it is d_2-bounded.

4. if $A \subseteq X$ is d_2-totally bounded, then it is d_1-totally bounded.

5. if there exists $c' > 0$ such that $c'd_2 \leq d_1 \leq cd_2$, then $A \subseteq X$ is d_1-totally bounded iff it is d_2-totally bounded.

Useful Theorems and Results

Diagonal of a Euclidean N-cube

Let C be a "cube" in a Euclidean space with side length $\alpha > 0$. That is, if $x \in \mathbb{R}^N$ is the "center" of C, then

$$C = \left[x_1 - \frac{\alpha}{2}, x_1 + \frac{\alpha}{2}\right] \times \left[x_2 - \frac{\alpha}{2}, x_2 + \frac{\alpha}{2}\right] \times ... \times \left[x_N - \frac{\alpha}{2}, x_N + \frac{\alpha}{2}\right]$$

Then the maximum distance from this center x is equal to

$$\begin{aligned}
\max_{y \in C} d(x, y) &= \max_{y \in C} \sqrt{\sum_{i=1}^{N} |x_i - y_i|^2} \\
&= \max_{\left\{y_i \in \left[x_i - \frac{\alpha}{2}, x_i + \frac{\alpha}{2}\right]\right\}_{i=1}^{N}} \sqrt{\sum_{i=1}^{N} |x_i - y_i|^2} \\
&= \sqrt{\sum_{i=1}^{N} \left|x_i - x_i \pm \frac{\alpha}{2}\right|^2} \\
&= \sqrt{\sum_{i=1}^{N} \left|\pm \frac{\alpha}{2}\right|^2} \\
&= \frac{\alpha}{2} \sqrt{\sum_{i=1}^{N} 1} \\
&= \frac{\alpha}{2} \sqrt{N}
\end{aligned}$$

and corresponds to all the "corners" of this N-cube.

Riesz's Lemma

For (X, d) normed vector space (i.e. the metric d is a p-norm), $(S, d|_{S \times S})$ non-dense linear subspace of (X, d), and $0 < \varepsilon < 1$, there exists $x \in X$ *of unit norm* (i.e. $d(\mathbf{0}, x) = ||x||_p = 1$) such that $d(x, s) \geq 1 - \varepsilon, \forall s \in S$.

Pigeonhole Principle

For $n, m, k \in \mathbb{N}$ with $n = km + 1$, if we distribute n elements across m sets then at least one set will contain at least $k + 1$ elements.