Athens University of Economics and Business
Department of Economics

Postgraduate Program - MSc in Economic Theory
Course: Mathematical Economics (Mathematics II)
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Solutions to Problem Set 1

Metric functions and metric spaces

Exercise 1

Is $d(x, y) = |x - y|$ a metric?

Metric spaces are defined by pairs of **non-empty sets** and **metric functions**.
A function can only be a metric function over a specified non-empty carrier set. No function can be a metric on its own merit. Here no such set is specified, so d is not a metric.
Furthermore, the formal way to define a function requires that its domain be specified. This is not the case here, so d is not even a properly defined function, to begin with.

Exercise 2

Is the function $d : X \times X \to \mathbb{R}$ such that $d(x, y) = |x - y|, \forall x, y \in X$ a metric on the non-empty set $X \subseteq \mathbb{R}$?

For d to be a suitable metric on the X, it needs to be a function such that $d : X \times X \to \mathbb{R}$ with X non-empty, which satisfies the following properties:

i) $d(x, y) \geq 0, \forall x, y \in X$ **(positivity)**

ii) $d(x, y) = 0 \iff x = y, \forall x, y \in X$ **(separateness)**

iii) $d(x, y) = d(y, x), \forall x, y \in X$ **(symmetry)**

iv) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$ (**subadditivity/triangle inequality**)

d is indeed a function such that $d : X \times X \to \mathbb{R}$ with X non-empty. So we need to test whether every property (i-iv) holds **for all elements of** X:

i) $d(x, y) = |x - y| \geq 0, \forall x, y \in X$

ii) $d(x, y) = 0 \iff |x - y| = 0 \iff x = y, \forall x, y \in X$

iii) $d(x, y) = |x - y| = |y - x| = d(y, x), \forall x, y \in X$

iv) $d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y| = d(x, z) + d(z, y), \forall x, y, z \in X$

(iv holds because of the triangle inequality for the real numbers)
So d is a suitable metric function on X.

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Exercise 3

Suppose that (Y, d) is a metric space. Let $f : X \to Y$ be an injection from X to Y. Define $d_f : X \times X \to \mathbb{R}$ such that $d_f(x, y) = d(f(x), f(y))$, $\forall x, y \in X$. Is (X, d_f) a metric space?

Since (Y, d) is a metric space, d is a metric on Y and satisfies the properties i-iv for all elements of Y.

f being injective means that every element of X is mapped onto an element of Y through f uniquely. No two elements of X have the same image on Y through f. Symbolically that means $f(x) = f(y) \Rightarrow x = y$, $\forall x, y \in X$. Naturally, also $x = y \Rightarrow f(x) = f(y)$, $\forall x, y \in X$. So $f(x) = f(y) \iff x = y$, $\forall x, y \in X$.

For the properties i-iv for the pair (X, d_f):

i) $d_f(x, y) = d(f(x), f(y)) \overset{i}{\geq} 0$, $\forall f(x), f(y) \in X$

ii) $d_f(x, y) = 0 \iff d(f(x), f(y)) = 0 \overset{ii}{\iff} f(x) = f(y) \iff x = y$, $\forall x, y \in X$

iii) $d_f(x, y) = d(f(x), f(y)) \overset{iii}{=} d(f(y), f(x)) = d_f(y, x)$, $\forall x, y \in X$

iv) $d_f(x, y) = d(f(x), f(y)) \overset{iv}{\leq} d(f(x), f(z)) + d(f(z), f(y)) = d_f(x, z) + d_f(z, y)$, $\forall x, y, z \in X$

So d_f is a suitable metric on X and (X, d_f) is a metric space.

Exercise 4

Study whether or not the following pairs of sets and functions constitute metric spaces:

1. $X \neq \varnothing$ and $d(x, y) = \begin{cases} 0, & x = y \\ c, & x \neq y \end{cases}$, $\forall x, y \in X$, with $c > 0$ (Discrete distance)

   It can be shown that (X, d) is indeed a metric space.

2. $X = \mathbb{R}$ and $d(x, y) = |e^x - e^y|$, $\forall x, y \in X$ [Sutherland Ex. 5.4 (b)]

   It can be shown that (X, d) is indeed a metric space.

3. $X = \varnothing$ and $d(x, y) = |x - y|$, $\forall x, y \in X$

   Generally, the properties of metric spaces with empty carrier sets is not of much interest. So formally we define metric spaces only for non-empty carrier sets. Here, X is empty, thus (X, d) cannot be a metric space.

   However, we could define a metric for the empty set.

   First, consider that the cartesian product between the empty set and any other set is the empty set, because the number of ordered pairs between elements of the two sets is exactly zero.

   $$\varnothing \times A = \varnothing$$

   Now consider the empty function that maps elements from the empty set to the empty set, $f_\varnothing : \varnothing \to \varnothing$. By the property of cartesian products involving empty sets and the fact that the empty set is a subset of any set, we can write $f_\varnothing : \varnothing \times \varnothing \to \mathbb{R}$.

   Furthermore, $f_\varnothing$ satisfies properties i-iv because there are no elements in its domain that violate them. So $(\varnothing, f_\varnothing)$ is a metric space. Pay attention to the fact that there is exactly one empty function, so the empty set can only be endowed with one metric.

Even more peculiarly, any function between any two sets, $f : A \to B$, can be thought of as a **subset of the cartesian products of its domain with its co-domain**, $A \times B$. So we can think of $f_{\varnothing}$ as $\varnothing \times \varnothing = \varnothing$. So, in this line of thought, $(\varnothing, \varnothing)$ is a metric space.

4. $X = \mathbb{R}$ and $d(x, y) = ln(|e^x - e^y|)$, $\forall x, y \in X$

   Since $x, y \in \mathbb{R}$ there exist x, y such that $|x - y| < 1 \Rightarrow ln(|x - y|) < 0$, so d does not satisfy property i on X and (X, d) is not a metric space. Even more so, $d(x, x)$ is not a real number.

5. $X = [-1, 1]$ and $d(x, y) = |x^2 - y^2|$, $\forall x, y \in X$

   Observe that $d(1, -1) = 0$ but $1 \neq -1$. So d does not satisfy property ii on X and (X, d) is not a metric space.

6. $X = \mathbb{R}$ and $d(x, y) = |x - y^3|$, $\forall x, y \in X$

   Let, for example, $x = 2$ and $y = 3$. Then $d(x, y) = 7$ but $d(y, x) = 1$. So d does not satisfy property iii on X and (X, d) is not a metric space.

7. $X = [0, 1]$ and $d(x, y) = |x - y|^2$, $\forall x, y \in X$

   Let $x = 0$, $y = 1$, and $z = \frac{1}{2}$. While for the usual metric on subsets of $\mathbb{R}$ (i.e. the absolute difference) the triangle inequality is obviously satisfied, this is not necessarily the case when we take its square. $d(x, y) = 1$, $d(x, z) = \frac{1}{4}$, and $d(y, z) = \frac{1}{4}$. So there exist $x, y, z \in X$ such that $d(x, y) > d(x, z) + d(y, z)$. So d does not satisfy property iv on X and (X, d) is not a metric space.

8. $X = \mathbb{R}^N$ and $d(x, y) = \left(\sum_{i=1}^{N} |x_i - y_i|^p\right)^{\frac{1}{p}}$, $\forall x, y \in X$, with $p, N \in \mathbb{N}^*$ (Minkowski distance)

   The Minkowski metric is a metric that generalizes many other metrics in normed vector spaces (such as the Manhattan metric for $p = 1$, the Euclidean metric for $p = 2$, and the Chebyshev metric as $p \to +\infty$).

   To prove that d is a metric on $\mathbb{R}^N$, we need to show that it satisfies the properties of metric functions, i-iv, on $\mathbb{R}^N$.

   i) For all $x, y \in \mathbb{R}^N$ and $x_i, y_i \in \mathbb{R}$ the i-th elements of x and y, respectively, we have

$$
\begin{aligned}
|x_i - y_i| \geq 0, \forall i \in \{1, 2, ..., N\} &\iff \\
|x_i - y_i|^p \geq 0, \forall i \in \{1, 2, ..., N\} &\iff \\
\sum_{i=1}^{N} |x_i - y_i|^p \geq 0 &\iff \\
\left(\sum_{i=1}^{N} |x_i - y_i|^p\right)^{\frac{1}{p}} \geq 0 &\iff \\
d(x, y) \geq 0
\end{aligned}
$$

ii) For all $x, y \in \mathbb{R}^N$

$$
\begin{aligned}
d(x,y) = 0 &\iff \\
\left(\sum_{i=1}^{N} |x_i - y_i|^p\right)^{\frac{1}{p}} = 0 &\iff \\
\sum_{i=1}^{N} |x_i - y_i|^p = 0 &\iff \\
|x_i - y_i|^p = 0, \forall i \in \{1, 2, ..., N\} &\iff \qquad \text{(sum of non-negative real numbers)} \\
|x_i - y_i| = 0, \forall i \in \{1, 2, ..., N\} &\iff \\
x_i = y_i, \qquad \forall i \in \{1, 2, ..., N\} &\iff \\
x = y &
\end{aligned}
$$

iii) $d(x,y) = \left(\sum_{i=1}^{N} |x_i - y_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{N} |y_i - x_i|^p\right)^{\frac{1}{p}} = d(y,x),\ \forall x, y \in \mathbb{R}^N$

iv) To show subadditivity we will employ Hölder's inequality. Because of the restriction on $\alpha \neq 1$ and $\beta \neq 1$ for Hölder's inequality to hold, we need to consider the case of $p = 1$ separately.

<u>Case: $p = 1$</u>

If $p = 1$, then $d(x,y) = \sum_{i=1}^{N} |x_i - y_i|,\ \forall x, y \in \mathbb{R}^N$ and subadditivity can be shown easily using the triangle inequality for the real numbers.

<u>Case: $p > 1$</u>

For any $x, y, z \in \mathbb{R}^N$ such that $x = y$ we want to show that

$$d(x,y) \leq d(x,z) + d(z,y) \overset{ii}{\iff} 0 \leq d(x,z) + d(z,y) \overset{i}{\iff} d(x,z) \geq 0 \text{ and } d(z,y) \geq 0$$

which trivially holds for all $z \in \mathbb{R}^N$.

For any $x, y, z \in \mathbb{R}^N$ such that $x \neq y$, consider the value of $d(x,y)$ raised to the power p

$$
\begin{aligned}
(d(x,y))^p &= \sum_{i=1}^{N} |x_i - y_i|^p \\
&= \sum_{i=1}^{N} |x_i - y_i||x_i - y_i|^{p-1} \\
&= \sum_{i=1}^{N} |x_i - z_i + z_i - y_i||x_i - y_i|^{p-1} \\
&\leq \sum_{i=1}^{N} (|x_i - z_i| + |z_i - y_i|)|x_i - y_i|^{p-1} \\
&= \sum_{i=1}^{N} |x_i - z_i||x_i - y_i|^{p-1} + \sum_{i=1}^{N} |z_i - y_i||x_i - y_i|^{p-1}
\end{aligned}
$$

We can apply Hölder's inequality for each of the two sums above. Choose $\alpha = p$ and find β as

$$\frac{1}{p} + \frac{1}{\beta} = 1 \iff \frac{1}{\beta} = 1 - \frac{1}{p} \iff \beta = \frac{1}{1 - \frac{1}{p}} \iff \beta = \frac{p}{p-1}$$

So now by Hölder's inequality we have

$$\begin{aligned}
(d(x,y))^p &\leq \left(\sum_{i=1}^{N} |x_i - z_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{N} \left(|x_i - y_i|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&+ \left(\sum_{i=1}^{N} |z_i - y_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{N} \left(|x_i - y_i|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&= \left(\left(\sum_{i=1}^{N} |x_i - z_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{N} |z_i - y_i|^p\right)^{\frac{1}{p}}\right) \left(\sum_{i=1}^{N} \left(|x_i - y_i|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&= \left(\left(\sum_{i=1}^{N} |x_i - z_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{N} |z_i - y_i|^p\right)^{\frac{1}{p}}\right) \left(\sum_{i=1}^{N} |x_i - y_i|^p\right)^{\frac{p-1}{p}} \\
&= \left(d(x,z) + d(z,y)\right) \left(d(x,y)\right)^{p-1}
\end{aligned}$$

Because $x \neq y \iff d(x,y) \neq 0$, by multiplying both sides by $(d(x,y))^{1-p}$ we get

$$d(x,y) \leq d(x,z) + d(z,y)$$

as required.

So (X,d) constitutes a metric space.

Exercise 5

For any metric space (X,d) and $\forall x,y,z,w \in X$, show that:

1. $|d(x,z) - d(z,y)| \leq d(x,y)$ [O'Searcoid Theorem 1.1.2, Sutherland Ex. 5.1]

   It holds for all $x,y,z \in X$ that

$$\begin{aligned}
|d(x,z) - d(z,y)| &\overset{iv}{\leq} |d(x,y) + d(y,z) - d(z,y)| \\
&\overset{iii}{=} |d(x,y)| \\
&\overset{i}{=} d(x,y)
\end{aligned}$$

2. $|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w)$ [O'Searcoid Q 1.2, Sutherland Ex. 5.2]

   For all $x,y,z,w \in X$ it holds that

$$\begin{aligned}
|d(x,y) - d(z,w)| &\overset{iv}{\leq} |d(x,z) + d(z,y) - d(z,w)| \\
&\overset{i}{\leq} d(x,z) + |d(z,y) - d(z,w)| \\
&\overset{iii,1}{\leq} d(x,z) + d(y,w)
\end{aligned}$$

Exercise 6

Let X be some non-empty set. Let d_1, d_2, and d_s be distance functions on X such that $d_s = d_1 + d_2$ Determine whether the following statements always hold (or under which conditions they could hold):

1. If d_1 and d_2 are metrics on X, d_s is a metric on X.

   i) $d_s(x,y) = d_1(x,y) + d_2(x,y) \overset{i}{\geq} 0,\ \forall x,y \in X$ as the sum of two non-negative values.

   ii) $d_s(x,y) = 0 \iff d_1(x,y) + d_2(x,y) = 0 \overset{i}{\iff} d_1(x,y) = 0$ and $d_2(x,y) = 0 \overset{ii}{\iff} x = y,\ \forall x,y \in X$

   iii) $d_s(x,y) = d_1(x,y) + d_2(x,y) \overset{iii}{=} d_1(y,x) + d_2(y,x) = d_s(y,x),\ \forall x,y \in X$

   iv) We have that for all $x,y,z, \in X$

   $$\begin{aligned} d_s(x,y) &= d_1(x,y) + d_2(x,y) \\ &\overset{iv}{\leq} d_1(x,z) + d_1(z,y) + d_2(x,z) + d_2(z,y) \\ &= d_1(x,z) + d_2(x,z) + d_1(z,y) + d_2(z,y) \\ &= d_s(x,z) + d_s(z,y) \end{aligned}$$

   So d_s is a metric on X.

2. If d_1 is a metric and d_2 a pseudo-metric on X, d_s is a metric on X.

   For i, iii, and iv see 1. For ii:

   Let $x,y \in X$ such that $x = y$. Then

   $$d_s(x,y) = d_1(x,y) + d_2(x,y) = 0 + 0 = 0$$

   Let $x,y \in X$ such that $x \neq y$. Then $d_1(x,y) > 0$ and $d_2(x,y) \geq 0$ thus

   $$d_s(x,y) = d_1(x,y) + d_2(x,y) > 0$$

   So d_s is a metric on X.

3. If d_1 and d_2 are pseudo-metrics on X, d_s is a metric on X.

   For i, iii, iv, see 1. For ii:

   For $x,y \in X$ such that $x = y$, the same logic as in 2. holds.

   For $x,y \in X$ such that $x \neq y$, if d_1 and d_2 are never simultaneously zero, then d_s is a metric on X. If not, then d_s is a pseudo-metric on X.

Exercise 7

Consider a finite index set $\mathcal{I} = \{1, 2, ..., n\}$ with $n \in \mathbb{N}^*$ and for each of its elements, i, the functional metric spaces $(\mathcal{B}(X_i, \mathbb{R}), d^i_{sup})$ with

$$d^i_{sup}(f_i, g_i) = \sup_{x \in X_i} |f_i(x) - g_i(x)|,\ \forall f_i, g_i \in \mathcal{B}(X_i, \mathbb{R})$$

Consider the product set $B_\Pi := \prod_{i\in I} \mathcal{B}(X_i, \mathbb{R})$ with $f := (f_i)_{i\in I} \in B_\Pi$ and the function $d_\Pi : B_\Pi \times B_\Pi \to \mathbb{R}$ such that

$$d_\Pi(f,g) = \max_{i\in\mathcal{I}} \sup_{x\in X_i} |f_i(x) - g_i(x)|, \, \forall f,g \in B_\Pi$$

Is (B_Π, d_Π) a metric space?

Since all $(\mathcal{B}(X_i,\mathbb{R}), d^i_{sup})$, $\forall i \in \mathcal{I}$ are metric spaces, all d^i_{sup} satisfy properties i-iv on $\mathcal{B}(X_i,\mathbb{R})$ for all i.
Notice that $d_\Pi(f,g) = \max_{i\in\mathcal{I}} \sup_{x\in X_i} |f_i(x) - g_i(x)| = \max_{i\in\mathcal{I}} d^i_{sup}(f_i, g_i)$. We will prove that this generalized case is a metric on B_Π irrespective of the functional form of the d^i_{sup} and thus drive the result.
Note that if $f \in B_\Pi$ then $f_i \in f \Rightarrow f_i \in \mathcal{B}(X_i,\mathbb{R})$, $\forall i \in \mathcal{I}$ (i.e. the i-th element of f always belongs to $\mathcal{B}(X_i,\mathbb{R})$). This guaranties that results that hold for elements of $\mathcal{B}(X_i,\mathbb{R})$ also hold for elements of f. So

i) Notice that for all elements of $f, g \in B_\Pi \Rightarrow f_i, g_i \in \mathcal{B}(X_i,\mathbb{R})$, $\forall i \in \mathcal{I}$ it holds that

$$\begin{aligned} d^i_{sup}(f_i, g_i) &\geq 0, \, \forall i \in \mathcal{I} \\ \max_{i\in\mathcal{I}} d^i_{sup}(f_i, g_i) &\geq 0 \\ d_\Pi(f,g) &\geq 0 \end{aligned}$$

ii) $f = g \iff f_i = g_i, \forall i \in \mathcal{I} \iff d^i_{sup}(f_i, g_i) = 0, \forall i \in \mathcal{I} \iff \max_{i\in\mathcal{I}} d^i_{sup}(f_i, g_i) = 0 \iff d_\Pi(f,g) = 0, \forall f,g \in B_\Pi$

iii) $d_\Pi(f,g) = \max_{i\in\mathcal{I}} d^i_{sup}(f_i, g_i) = \max_{i\in\mathcal{I}} d^i_{sup}(g_i, f_i) = d_\Pi(g,f), \forall f,g \in B_\Pi$

iv) Let $f, g, h \in B_\Pi \Rightarrow f_i, g_i, h_i \in \mathcal{B}(X_i,\mathbb{R})$, $\forall i \in \mathcal{I}$, then

$$\begin{aligned} d_\Pi(f,g) &= \max_{i\in\mathcal{I}} d^i_{sup}(f_i, g_i) \\ &\leq \max_{i\in\mathcal{I}} \{d^i_{sup}(f_i, h_i) + d^i_{sup}(h_i, g_i)\} \\ &\leq \max_{i\in\mathcal{I}} d^i_{sup}(f_i, h_i) + \max_{i\in\mathcal{I}} d^i_{sup}(h_i, g_i) \\ &= d_\Pi(f,h) + d_\Pi(h,g) \end{aligned}$$

Exercise 8 [O'Searcoid Q 1.8]
Let $P(S)$ be the power set of a non empty set, S. Let the function $d : P(S) \times P(S) \to \mathbb{R}$ such that

$$d(A,B) = |(A \setminus B) \cup (B \setminus A)|, \, \forall A, B \in P(S)$$

be a function that gives the cardinality of the symmetric difference between two elements of $P(S)$ (i.e. subsets of S). Is d a metric on $P(S)$?

i) By the definition of cardinality $d(A,B) = |(A \setminus B) \cup (B \setminus A)| \geq 0$, $\forall A, B \in P(S)$.

ii) Remember that the empty set has zero elements. Thus, its cardinality is equal to zero (and, of course, no non-empty set can have zero cardinality).

Let $A, B \in P(S)$ with $A = B$, then $A \setminus B = B \setminus A = \varnothing$ and $d(A, B) = 0$.

Let $A, B \in P(S)$ with $A \neq B$, then $A \setminus B \neq \varnothing$ or $B \setminus A \neq \varnothing$ and $d(A, B) \neq 0$.

iii) $d(A, B) = |(A \setminus B) \cup (B \setminus A)| = |(B \setminus A) \cup (A \setminus B)| = d(B, A),\ \forall A, B \in P(S)$

iv) Let A, B, and C be any subsets of S (thus $A, B, C \in P(S)$). Then,

$$|A \setminus B| = |A| - |A \cap B|$$

and

$$|A \setminus B| \cup |B \setminus A| = |A| - |A \cap B| + |B| - |B \cap A| = |A| + |B| - 2|A \cap B|$$

Similarly

$$|A \setminus C| \cup |C \setminus A| = |A| + |C| - 2|A \cap C|$$

and

$$|B \setminus C| \cup |C \setminus B| = |B| + |C| - 2|B \cap C|$$

And we want to show that for all $A, B, C \in P(S)$

$$\begin{aligned}
d(A, B) &\leq d(A, C) + d(B, C) \\
|A \setminus B| \cup |B \setminus A| &\leq |A \setminus C| \cup |C \setminus A| + |B \setminus C| \cup |C \setminus B| \\
|A| + |B| - 2|A \cap B| &\leq |A| + |C| - 2|A \cap C| + |B| + |C| - 2|B \cap C| \\
0 &\leq 2|C| + 2|A \cap B| - 2|A \cap C| - 2|B \cap C| \\
0 &\leq |C| + |A \cap B| - |A \cap C| - |B \cap C| \\
0 &\leq |C| + |(A \cap B) \setminus C| + |A \cap B \cap C| - |(A \cap C) \setminus B| - |A \cap B \cap C| - |(B \cap C) \setminus A| - |A \cap B \cap C| \\
0 &\leq |C| + |(A \cap B) \setminus C| - |(A \cap C) \setminus B| - |(B \cap C) \setminus A| - |A \cap B \cap C| \\
0 &\leq |C| - |(A \cap C) \setminus B| - |(B \cap C) \setminus A| + |(A \cap B) \setminus C| - |A \cap B \cap C| \\
0 &\leq |C \setminus (A \cup B)| + |A \cap B \cap C| + |(A \cap B) \setminus C| - |A \cap B \cap C| \\
0 &\leq |C \setminus (A \cup B)| + |(A \cap B) \setminus C|
\end{aligned}$$

which always holds as the sum of non-negative values.

(Notice that: $|C|$ is the number of elements in C. $|(A \cap C) \setminus B|$ is the number of elements of C that are also in A except for those that are also in B. $|(B \cap C) \setminus A|$ is the number of elements of C that are also in B except for those that are also in A. $|C| - |(A \cap C) \setminus B| - |(B \cap C) \setminus A|$ is the number of elements in C, less the number of its elements that are also in A but not B, less the number of its elements that are also in B but not A. So from C we are removing almost all elements in $A \cap C$ and $B \cap C$, except for those in $A \cap B \cap C$. This is equivalent to removing from C all elements in A and/or B and then adding back the elements in $A \cap B \cap C$.

So $|C| - |(A \cap C) \setminus B| - |(B \cap C) \setminus A| = |C \setminus (A \cup B)| + |A \cap B \cap C|$.)

So d is a metric on $P(S)$.

Exercise 9 [Sutherland Ex. 5.14]

Let n be a positive natural number. The distance functions:

1. $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$, $\forall x, y \in \mathbb{R}^n$ (Manhattan distance)
2. $d_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $d_2(x, y) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$, $\forall x, y \in \mathbb{R}^n$ (Euclidean distance)
3. $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $d_\infty(x, y) = \max_{i=1}^n |x_i - y_i|$, $\forall x, y \in \mathbb{R}^n$ (Chebyshev distance)

are all metrics on $\mathbb{R}^n$. Show that the following functional inequalities hold:

$$d_\infty \leq d_2 \leq d_1 \leq n \cdot d_\infty \leq n \cdot d_2 \leq n \cdot d_1$$

Let x and y be arbitrary elements of $\mathbb{R}^n$.

We will start with $d_\infty \leq d_2$ (and consequently $n \cdot d_\infty \leq n \cdot d_2$):

Observe that by taking the square of d_∞ we get

$$d_\infty^2(x, y) = \left(\max_{i=1}^n |x_i - y_i|\right)^2 = \max_{i=1}^n |x_i - y_i|^2$$

By squaring d_2 we get

$$d_2^2(x, y) = \left(\left(\sum_{i=1}^n |x_i - y_i|^2\right)^{\frac{1}{2}}\right)^2 = \sum_{i=1}^n |x_i - y_i|^2$$

Obviously the greatest among the $|x_i - y_i|^2$ is among the summed (non-negative) elements, thus naturally

$$\max_{i=1}^n |x_i - y_i|^2 \leq \sum_{i=1}^n |x_i - y_i|^2$$
$$d_\infty^2(x, y) \leq d_2^2(x, y)$$
$$d_\infty(x, y) \leq d_2(x, y)$$

because squaring is an affine transformation.

So $d_\infty \leq d_2$ (and $n \cdot d_\infty \leq n \cdot d_2$).

We proceed with $d_2 \leq d_1$ (and consequently $n \cdot d_2 \leq n \cdot d_1$):

Again, we square both metric and get

$$d_2^2(x, y) = \sum_{i=1}^n |x_i - y_i|^2$$

and

$$d_1^2(x, y) = \left(\sum_{i=1}^n |x_i - y_i|\right)^2$$

Observe that $d_1^2(x,y)$ is the square of the sum of n non-negative real numbers, while $d_2^2(x,y)$ is the sum of those same numbers squared. Thus,

$$\begin{aligned} d_1^2(x,y) &= \left(\sum_{i=1}^{n} |x_i - y_i|\right)^2 \\ &= \sum_{i=1}^{n} |x_i - y_i|^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} |x_i - y_i||x_j - y_j| \\ &= d_2^2(x,y) + 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} |x_i - y_i||x_j - y_j| \end{aligned}$$

where the trailing sum is positive.

So $d_2 \leq d_1$ (and $n \cdot d_2 \leq n \cdot d_1$), which also means that $d_\infty \leq d_2 \leq d_1$ (and $n \cdot d_\infty \leq n \cdot d_2 \leq n \cdot d_1$).

Finally, consider summing up d_∞ n times. Then

$$\sum_{i=1}^{n} d_\infty(x,y) = \sum_{i=1}^{n} \max_{i=1}^{n} |x_i - y_i|$$

Naturally this sum cannot be smaller than the plain sum of all $|x_i - y_i|$

$$\begin{aligned} \sum_{i=1}^{n} |x_i - y_i| &\leq \sum_{i=1}^{n} \max_{i=1}^{n} |x_i - y_i| \\ \sum_{i=1}^{n} |x_i - y_i| &\leq n \cdot \max_{i=1}^{n} |x_i - y_i| \\ d_1(x,y) &\leq n \cdot d_\infty(x,y) \end{aligned}$$

So $d_1 \leq n \cdot d_\infty$.

Thus, we have proven that

$$d_\infty \leq d_2 \leq d_1 \leq n \cdot d_\infty \leq n \cdot d_2 \leq n \cdot d_1$$

Exercise 10

Let X be an $n \times m$ real matrix, with $n, m \in \mathbb{N}^*$ and $n > m$, such that $rank(X) = m$. Then $P_X = X(X'X)^{-1}X'$ is the projection matrix of X. Let $Y \subseteq \mathbb{R}^n$ be non-empty and $\hat{Y}$ be its projected image through P_X. Define $d_X : Y \times Y \to \mathbb{R}$ such that $d_X(x,y) = ||P_X \cdot x - P_X \cdot y||$, $\forall x, y \in Y$ (i.e. d_X is the Euclidean norm of an n-dimensional real vector). Show that (Y, d_X) is a pseudo-metric space.

(Hint: Consider the example of exercise 3. Under which conditions for f is (X, d_f) a pseudo-metric space?)

Let d be the appropriate Euclidean metric on $\hat{Y}$. Define $f : Y \to \hat{Y}$ as $f(x) = P_X x$, $\forall x \in Y$. We will consider d_X as the composition $d_X(\cdot,\cdot) = d(f(\cdot), f(\cdot))$.

In exercise 3 we basically showed that for some metric space (Y, d) and some injective function $f : X \to Y$, the composition $d_f(\cdot,\cdot) = d(f(\cdot), f(\cdot))$ is a metric on X.

Observe that the properties i, iii, iv hold as long as f is a properly defined function from X to Y. The injective property is only needed for ii. Further notice, however, that as long as f is thus well defined

$$x = y \Rightarrow f(x) = f(y) \iff d(f(x), f(y)) = 0$$

for all such x and y in X, while for any $x, y \in X$ such that $d(f(x), f(y)) = 0$ it doesn't necessarily follow that $x = y$. So if f is a well defined function $f : X \to Y$, d_f is a pseudo-metric on X.
Here we have that $(\hat{Y}, d)$ is a metric space and are given a function $f : Y \to \hat{Y}$ such that $d_X(\cdot, \cdot) = d(f(\cdot), f(\cdot)) : Y \times Y \to \mathbb{R}$. By the above, we get that d_X is a pseudo-metric on Y and (Y, d_X) a pseudo-metric space.

Exercise 11

Let (X, d) be a metric space and consider a real function $f : \mathbb{R} \to \mathbb{R}$. Define $d' : X \times X \to \mathbb{R}$ such that $d'(x, y) = f(d(x, y))$, $\forall x, y \in X$.

1. Deduce the necessary conditions for f so that d' be a metric on X.

   d' has to satisfy properties i-iv on X:

   i) For positivity we want $\forall x, y \in X$

   $$d'(x, y) \geq 0 \iff f(d(x, y)) \geq 0$$

   and notice that $x, y \in X \Rightarrow d(x, y) \geq 0$. So for positivity we need $x \geq 0 \Rightarrow f(x) \geq 0$.

   ii) For separateness we want:

   $$\begin{aligned} d'(x, y) = 0 &\iff x = y, \forall x, y \in X \\ f(d(x, y)) = 0 &\iff x = y, \forall x, y \in X \\ f(d(x, y)) = 0 &\iff d(x, y) = 0 \end{aligned}$$

   So for separateness we need $f(x) = 0 \iff x = 0$.

   iii) It naturally holds that $d'(x, y) = f(d(x, y)) = f(d(y, x)) = d'(y, x)$, $\forall x, y \in X$. So no extra condition needs to hold for symmetry.

   iv) For the triangle inequality (aslo called subadditivity) we want:

   $$\begin{aligned} d'(x, y) &\leq d'(x, z) + d'(z, y), && \forall x, y, z \in X \\ f(d(x, y)) &\leq f(d(x, z)) + f(d(z, y)), && \forall x, y, z \in X \end{aligned}$$

   Don't forget that it also always holds that $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$. When this holds with equality (i.e. when $d(x, y) = d(x, z) + d(z, y)$) the desired property for f is called subadditivity and is defined as

   $$f(x + y) \leq f(x) + f(y), \forall x, y \in \mathbb{R}$$

So f has to be subadditive.

For the cases when $d(x,y) < d(x,z) + d(z,y)$, let's consider $z < x + y$ and notice that if $f(z) \leq f(x+y)$, by subadditivity we get

$$f(z) \leq f(x+y) \leq f(x) + f(y)$$

which is the desired property. So (weak) monotonicity is also necessary.

So f has to be (weakly) increasing and subadditive.

Finally, notice that monotonicity and $f(0) = 0$ already guarantee that $x \geq 0 \Rightarrow f(x) \geq 0$.

So to summarize, for d' to be a metric on X, the necessary conditions for f are that f be a weakly increasing subadditive function with $f(0) = 0$.

2. Let $f(x) = \dfrac{x}{1+x}$, $\forall x \in \mathbb{R}_+$. Is d' a metric on X?

3. Let $f(x) = ln(1+x)$, $\forall x \in \mathbb{R}_+$. Is d' a metric on X?

4. Let $f(x) = x^\alpha$, $\forall x \in \mathbb{R}_+$ with $0 < \alpha < 1$. Is d' a metric on X?

5. Let f be a strictly increasing concave real function such that $f(0) = 0$. Is d' a metric on X?

   Yes, because strict monotonicity also implies weak monotonicity and concave functions that take the value of zero when evaluated at zero are also subadditive.

   (Additionally, if we have a strictly increasing concave real function, f', such that $f'(0) \neq 0$, we can define f on the same domain such that $f(x) = f'(x) - f'(0)$.)

Useful Theorems and Results

Cardinality and Set Operations

Cardinality is a measure of the number of elements in a set. The following properties hold with respect to cardinality:

$$|\varnothing| = 0 \tag{1}$$

$$|A| + |B| = |A \cup B| + |A \cap B| \tag{2}$$

$$|A \setminus B| = |A| - |A \cap B| \tag{3}$$

Square of the sum of N numbers

$$\left(\sum_{i=1}^{N} a_i\right)^2 = \sum_{i=1}^{N} a_i^2 + 2\sum_{i=1}^{N}\sum_{j=1}^{i-1} a_i a_j \tag{4}$$

Hölder's inequality

For all $x, y \in \mathbb{R}^N$ and $\alpha, \beta \in (1, +\infty)$ such that $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$, it holds that

$$\sum_{i=1}^{N} |x_i y_i| \leq \left(\sum_{i=1}^{N} |x_i|^\alpha\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{N} |y_i|^\beta\right)^{\frac{1}{\beta}} \tag{5}$$

For $\alpha = \beta = 2$ we get the Cauchy-Schwartz inequality.

For $x \in \mathbb{R}^N$ we call $||x||_p \coloneqq \left(\sum_{i=1}^{N} |x_i|^p\right)^{\frac{1}{p}}$ the p-norm of x.