TA Session 5

Definition: Lipschitz continuity

Let (X, d_X) and (Y, d_Y) metric spaces, and $f: X \to Y$.
f is d_Y/d_X-Lipschitz continuous iff

$$\exists c > 0 : \forall x, y \in X \quad d_Y(f(x), f(y)) \leq c \cdot d_X(x, y)$$

Definition: Contractivity

Let (X, d) metric space and $f: X \to X$ a self map.
f is a d-contraction iff

$$\exists\, 0 < c < 1 : \forall x, y \in X \quad d(f(x), f(y)) \leq c \cdot d(x, y)$$

Lemmata:

- d-contractivity $\Rightarrow$ d/d-Lipschitz continuity
- d_Y/d_X-Lipschitz continuity $\Rightarrow$ d_Y/d_X-continuity

Also:

- continuous functions preserve sequential convergence between spaces
- Lipschitz continuous functions preserve Cauchyness between spaces

Definition: Vector Norms

Let $x \in \mathbb{R}^n$ be an n-dimensional real vector. The p-norm of x, with $p \in \mathbb{N}^*$, is defined as

$$\|x\|_p := \left(\sum_{i=1}^{n} x_i^p \right)^{1/p}$$

Definition: Frobenius Norm

Let $A \in \mathbb{R}^{n\times m}$ be an $n\times m$ real matrix. The Frobenius norm of A is defined as

$$\|A\|_F := \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2}$$

Lemma: Submultiplicativity of the Euclidean and the Frobenius norm

Let $x \in \mathbb{R}^m$ an m-dimensional column vector and $A \in \mathbb{R}^{n\times m}$ an $n\times m$ matrix. Then,

$$\underbrace{\|A\cdot x\|_2}_{\substack{\text{Euclidean norm} \\ \text{in } \mathbb{R}^n}} \leq \underbrace{\|A\|_F}_{\substack{\text{Frobenius norm} \\ \text{in } \mathbb{R}^{n\times m}}} \cdot \underbrace{\|x\|_2}_{\substack{\text{Euclidean norm} \\ \text{in } \mathbb{R}^m}}$$

Example

Let $(\mathbb{R}^n, d_n)$ and $(\mathbb{R}^m, d_m)$ Euclidean spaces, with d_n the Euclidean metric on $\mathbb{R}^n$ and d_m the Euclidean metric on $\mathbb{R}^m$.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be everywhere differentiable with a bounded derivative, i.e.,

$$\forall x \in \mathbb{R}^n \ \exists \frac{\partial f}{\partial x'}(x) \in \mathbb{R}^{m \times n} \text{ with } \left\| \frac{\partial f}{\partial x'}(x) \right\|_F < \infty$$

Then, f is d_m/d_n-Lipschitz continuous on $\mathbb{R}^n$.

Proof

By the Mean Value Theorem it holds that

$$f(x) = f(y) + \frac{\partial f}{\partial x'}(z) \cdot (x-y)$$

with $z = \lambda x + (1-\lambda)y$, $0 \leq \lambda \leq 1$ $\forall x, y \in \mathbb{R}^n$. So,

$$f(x) - f(y) = \frac{\partial f}{\partial x'}(z)\,(x-y) \Rightarrow$$

$$\Rightarrow \|f(x) - f(y)\|_2 = \left\| \frac{\partial f}{\partial x'}(z)\,(x-y) \right\|_2 \leq$$

$$\leq \left\| \frac{\partial f}{\partial x'}(z) \right\|_F \cdot \|x-y\|_2 \leq$$

$$\leq \sup_{z \in \mathbb{R}^n} \left\| \frac{\partial f}{\partial x'}(z) \right\|_F \cdot \|x-y\|_2$$

But, we can write

$$d_m(f(x), f(y)) \leq \sup_{z \in \mathbb{R}^n} \left\| \frac{\partial f}{\partial x'}(z) \right\|_F \cdot d_n(x, y) \quad \forall x, y \in \mathbb{R}^n$$

and by setting $c := \sup_{z \in \mathbb{R}^n} \left\| \frac{\partial f}{\partial x'}(z) \right\|_F > 0$

$$d_m(f(x), f(y)) \leq c \cdot d_n(x, y)$$

and f is d_m/d_n-Lipschitz continuous on $\mathbb{R}^n$ with Lipschitz coefficient $c = \sup_{z \in \mathbb{R}^n} \left\| \frac{\partial f}{\partial x'}(z) \right\|_F$

$\boxtimes$

Remark: The converse (almost) also holds. If $(\mathbb{R}^n, d_n)$, $(\mathbb{R}^m, d_m)$ Euclidean spaces and $f: \mathbb{R}^n \to \mathbb{R}^m$ is d_m/d_n-Lipschitz continuous, then it is differentiable with bounded derivative <u>almost everywhere</u> on $\mathbb{R}^n$.

Banach Fixed Point Theorem

Let:

- (X, d) be a complete metric space, and
- $f: X \to X$ a d-contraction.

Then:

- f has a unique fixed point, x_f, and furthermore
- $x_f = d\text{-}\lim\limits_{m \to +\infty} f^{(m)}(x)$, $\forall x \in X$.

Sketch of Proof

For some $x \in X$ denote $x_m = f^{(m)}(x) = \begin{cases} x & , m=0 \\ f(x_{m-1}) & , m \geq 0 \end{cases}$

You need to prove that:

- $(x_m)_{m \in \mathbb{N}}$ is d-Cauchy, by showing that $\forall m \geq 0 \quad d(x_{m+1}, x_m) \leq c^m \; d(x_1, x_0)$, $0 < c < 1$ via Mathematical Induction.

  Then, since (X,d) complete, $(x_m)_{m \in \mathbb{N}}$ d-convergent.
- If $(x_m)_{m \in \mathbb{N}}$ d-converges, then its d-limit is a fixed point of f, using the definition of x_f and the ~~d~~ d-continuity of f (due to d-contraction).
- If f has a fixed point, then it is unique, by assuming that it is not and using the d-contractiveness of f to arrive at a contradiction.