Math. Econ. II - TA 3

Definition: d-openness

Let (X, d) m.s. and $A \subseteq X$. A is termed a d-open subset of X iff

$$\forall x \in A \ \exists \varepsilon > 0 : \mathcal{O}_d(x, \varepsilon) \subseteq A$$

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$$A' := X \setminus A \text{ is d-open}$$

Examples in (X, d)

X is d-open since $\forall x \in X \ \exists \varepsilon > 0 : \mathcal{O}_d(x, \varepsilon) \subseteq X$

$\emptyset$ is d-open since it has no elements that violate the definition.

X is d-closed since $X' \equiv \emptyset$ which is d-open.

$\emptyset$ is d-closed since $\emptyset' \equiv X$ which is d-open.

Any $\mathcal{O}_d(x, \varepsilon)$ is d-open (see PS 2 Exercise 3).

Any $\mathcal{O}_d[x, \varepsilon]$ is d-closed (see notes on topologies).

Furthermore: (see notes on topologies)

- arbitrary unions of d-open sets are d-open.
- finite intersections of d-open sets are d-open.
- finite unions of d-closed sets are d-closed.
- arbitrary intersections of d-closed sets are d-closed.

Remark: The above motivate the desired properties that are included in the definition of what constitutes a topology on a carrier set, X.

Example in $(\mathbb{R}, d_u)$

$(0,1]$ is neither d_u-open, nor d_u-closed in $\mathbb{R}$.

Proof

$$\nexists\, \varepsilon > 0 : \mathcal{O}_{d_u}(1, \varepsilon) \subseteq (0,1]$$

$$\nexists\, \varepsilon > 0 : \mathcal{O}_{d_u}(0, \varepsilon) \subseteq (-\infty, 0] \cup (1, +\infty) = (0,1]' \quad \boxtimes$$

Exercises:

- Show that $(0,1)$ is a d_u-open subset of $\mathbb{R}$.
- Show that $[0,1]$ is a d_u-closed subset of $\mathbb{R}$.

Remark:

Notice that $[0,1]$ is d_u-open in $[0,1]$.

Definition: Sequence

Let $X \neq \emptyset$, then $(x_n)_{n \in \mathbb{N}}$ with

$$x_n \in (x_n)_{n \in \mathbb{N}} : x_n \in X \;\; \forall n \in \mathbb{N}$$

is termed an X-valued sequence.

A sequence can be thought of as a vector which has a first element, but no last element, and has the same number of elements as the set of natural numbers, $\mathbb{N}$.

Equivalently, a sequence can be thought of as a function from the natural numbers to the set in question, $f : \mathbb{N} \to X$.

(We can also say that $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$, which relates to the function characterization above.)

Definition: Sequential Convergence

Let (X, d) m.s. and $(x_n)_{n \in \mathbb{N}}$ an X-valued sequence. $x \in X$ is termed a d-limit of $(x_n)_{n \in \mathbb{N}}$ iff

$$\forall \varepsilon > 0 \;\; \exists n_\varepsilon \in \mathbb{N} : x_n \in \mathcal{O}_d(x, \varepsilon) \;\; \forall n \geq n_\varepsilon$$

and we can write $x = d\text{-}\lim(x_n)$, or equivalently $x_n \xrightarrow{d} x$.

If a sequence has a d-limit, it is called d-convergent. If it has no d-limit, it is called d-divergent.

Lemma: Limit Uniqueness in Metric Spaces

If a sequence, $(x_n)_{n\in\mathbb{N}}$, in a metric space, (X,d), has a d-limit, then that limit is unique.

Proof:

Let $x_n \xrightarrow{d} x$, $x_n \xrightarrow{d} y$, with $x \neq y$. Then, by the separateness property $d(x,y) > 0$. Thus, $\exists \varepsilon_x > 0, \varepsilon_y > 0$ s.t. $\varepsilon_x + \varepsilon_y < d(x,y)$ and $\mathcal{O}_d(x,\varepsilon_x) \cap \mathcal{O}_d(y,\varepsilon_y) = \emptyset$.

Since $x = d\text{-}\lim(x_n)$ almost all elements of $(x_n)_{n\in\mathbb{N}}$ lie in $\mathcal{O}_d(x,\varepsilon_x)$. This leaves only a finite number of elements that may lie in $\mathcal{O}_d(y,\varepsilon_y)$. But $y = d\text{-}\lim(x_n)$. Contradiction! $\boxtimes$

Remark: This is not necessarily true for pseudo-metric spaces.

Example in $(\mathbb{R}, d_u)$

Let $(x_n)_{n\in\mathbb{N}}$ be a real-valued sequence s.t. $x_n = \frac{1}{1+n} \in \mathbb{R}$. $\forall n \in \mathbb{N}$. Then, $x_n \xrightarrow{d_u} 0$.

Proof:

For some $\varepsilon > 0$ let $n_\varepsilon := \underset{n\in\mathbb{N}}{\operatorname{argmin}} \{ n : \frac{1}{1+n} < \varepsilon \}$.

Notice that $\exists n_\varepsilon \in \mathbb{N}$ $\forall \varepsilon > 0$. Consider $\mathcal{O}_{d_u}(0,\varepsilon) =$
$= \{ x \in \mathbb{R} : d(x,0) < \varepsilon \}$. Then $x_{n_\varepsilon} = \frac{1}{1+n_\varepsilon} \in \mathcal{O}_{d_u}(0,\varepsilon)$ since $d_u(0, x_{n_\varepsilon}) = \left| \frac{1}{1+n_\varepsilon} - 0 \right| = \frac{1}{1+n_\varepsilon} < \varepsilon$.

But also $d(x_m, 0) = \frac{1}{1+m} < \varepsilon \ \forall m \geq n_\varepsilon$ and there exist infinitely many such $m \in \mathbb{N}$.

Thus, at least almost all of the elements of $(x_n)_{n\in\mathbb{N}}$ lie inside $\mathcal{O}_{d_u}(0, \varepsilon)$, and $\varepsilon > 0$ was chosen arbitrarily, so this holds $\forall \varepsilon > 0$.

So $0 = d_u\text{-}\lim(x_n)$. $\boxtimes$

Definition: Eventually Constant Sequences

Let $(x_n)_{n\in\mathbb{N}}$ be some X-valued sequence s.t.

$(x_n)_{n\in\mathbb{N}} = (x_0, x_1, x_2, \ldots, x_{m-1}, c, c, c, \ldots)$

i.e., $x_n = c \ \forall n \geq m \in \mathbb{N}$. Then $(x_n)_{n\in\mathbb{N}}$ is called an eventually constant sequence.

Lemma: Convergence in Discrete Spaces

Let (X, d_c) be a discrete metric space. Then, only eventually constant X-valued sequencies have a d_c-limit.

Proof:

Let $(x_n)_{n\in\mathbb{N}}$ not be eventually constant, but $x_n \xrightarrow{d_c} x$.

Then $\forall \varepsilon > 0 \ \exists n_\varepsilon : x_n \in \mathcal{O}_{d_c}(x, \varepsilon) \ \forall n \geq n_\varepsilon$. But

$x_n \in \mathcal{O}_{d_c}(x, \varepsilon) = \{x_n \in X : d(x, x_n) < \varepsilon\} = \{x_n \in X : c < \varepsilon\}$

Choose $\varepsilon < \mathcal{E}$ and we arrive at a contradiction. $\boxtimes$

Lemma: d-convergence and d-boundness

Let (X, d) m.s. and $(x_n)_{n\in\mathbb{N}}$ be d-convergent. Then $(x_n)_{n\in\mathbb{N}}$ is a d-bounded subset of X.

Proof:

Let $x_n \xrightarrow{d} x$, then $\forall \varepsilon > 0\ \exists n_\varepsilon : x_n \in \mathcal{O}_d(x, \varepsilon)\ \forall n \geq n_\varepsilon$.

$\exists\ \varepsilon^* > 0 : n_{\varepsilon^*} = 0$, thus $x_n \in \mathcal{O}_d(x, \varepsilon^*)\ \forall n \geq 0$ or equivalently $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{O}_d(x, \varepsilon^*)$.

So $(x_n)_{n\in\mathbb{N}}$ is d-bounded. $\boxtimes$

Exercise:

Show that d-convergent sequences in metric spaces are d-totally bounded.

Remark:

The converse to the above does not always hold. That is, d-bounded and d-totally bounded sequences are not always d-convergent.

Lemma: Sequential Convergence and Metrics Comparison

Let $X \neq \emptyset$ and d_1, d_2 metrics on X s.t.

$$\exists\, c>0 : d_1 \leq c d_2$$

then $\left(x_n \xrightarrow{d_2} x\right) \Rightarrow \left(x_n \xrightarrow{d_1} x\right)$

Sketch of Proof:

Remember that

$$d_1 \leq c\, d_2 \Rightarrow \mathcal{O}_{d_2}(x,\varepsilon) \subseteq \mathcal{O}_{d_1}(x, c\cdot\varepsilon) \quad \forall \varepsilon>0, \forall x_0 \in X$$

You can use this to prove the Lemma's statement

Lemma: Sequential Characterization of Closedness

Let (X,d) m.s. and $A \subseteq X$. A is a d-closed subset of X iff

$$\left(\forall (x_n)_{n\in\mathbb{N}} \in A,\ \exists x : x_n \xrightarrow{d} x\right) \Rightarrow (x \in A)$$

Proof:

(see notes on topologies)

Definition: "Point-wise" Continuity of Functions Between Metric Spaces

Let (X, d_X), (Y, d_Y) m.s., $f: X \to Y$, and $x \in X$.

f is called d_Y/d_X-continuous at x iff

$$\forall (x_n)_{n\in\mathbb{N}}: x_n \xrightarrow{d_X} x \Rightarrow (f(x_n))_{n\in\mathbb{N}} \xrightarrow{d_Y} f(x)$$

Definition: Continuity of Functions Between M.S.

Let (X, d_X), (Y, d_Y) m.s., and $f: X \to Y$.

f is a d_Y/d_X-continuous function iff it is d_Y/d_X-continuous at all $x \in X$.

Example with Discret Domain

Let (X, d_c) discrete m.s. and (Y, d) general m.s. Notice that only eventually constant $(x_n)_{n\in\mathbb{N}}$ in X are d_c-convergent. But then $(f(x_n))_{n\in\mathbb{N}}$ are also eventually constant and converge in Y for any metric, d (trivialy). So for any function between X and Y convergence is preserved. So all functions with discrete domains are d/d_c-continuous.