

Completeness and Function Spaces

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Lemma

Let (Y, d) be a complete metric space and $X \neq \emptyset$ a non-empty set, then the structured set $(\mathcal{B}(X, Y), d_{sup}^d)$ with $d_{sup}^d(f, g) = \sup_{x \in X} d(f(x), g(x)), \forall f, g \in \mathcal{B}(X, Y)$ is a complete metric space.

Proof

We need to show that:

1. $(\mathcal{B}(X, Y), d_{sup}^d)$ is a metric space, which requires that:
 - $\mathcal{B}(X, Y)$ be a non-empty set.
 - d_{sup}^d be a metric function on $\mathcal{B}(X, Y)$ (properties i-iv).
2. Every d_{sup}^d -Cauchy sequence on $\mathcal{B}(X, Y)$ has a d_{sup}^d -limit in $\mathcal{B}(X, Y)$, which requires that for all $(f_n)_{n \in \mathbb{N}} : f_n \in \mathcal{B}(X, Y), \forall n \in \mathbb{N}$, there exists a function, f , such that:
 - $f = d_{sup}^d - \lim f_n$
 - $f \in \mathcal{B}(X, Y)$

1. Since $X \neq \emptyset$ we can define at least one function (if not more) that maps elements of X to elements of Y (also non-empty). For example, the constant function $f_c : X \rightarrow Y$ such that $\forall x \in X, f_c(x) = y_c$ for some $y_c \in Y$.

Furthermore, observe that $f_c(X) = \{y_c\} \subseteq Y$, i.e. the image of X through f_c is a subset of Y (naturally) and it is also a singleton set (it has only one element). Thus $f(X)$ is certainly a d -bounded subset of Y .

We have found at least one example of a bounded function from X to Y . So $\mathcal{B}(X, Y)$ is non-empty.

Also, d_{sup}^d is a metric function on $\mathcal{B}(X, Y)$ since:

- i) $\forall f, g \in \mathcal{B}(X, Y), d_{sup}^d(f, g) = \sup_{x \in X} d(f(x), g(x)) \geq 0$
- ii) $\forall f, g \in \mathcal{B}(X, Y),$

$$\begin{aligned}
 d_{sup}^d(f, g) = 0 & \iff \\
 \sup_{x \in X} d(f(x), g(x)) = 0 & \iff \\
 d(f(x), g(x)) = 0, \forall x \in X & \iff \\
 f(x) = g(x), \forall x \in X & \iff \\
 f = g &
 \end{aligned}$$

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$$\text{iii) } \forall f, g \in \mathcal{B}(X, Y), d_{sup}^d(f, g) = \sup_{x \in X} d(f(x), g(x)) \stackrel{\text{iii}}{=} \sup_{x \in X} d(g(x), f(x)) = d_{sup}^d(g, f)$$

$$\text{iv) } \forall f, g, h \in \mathcal{B}(X, Y),$$

$$\begin{aligned} d_{sup}^d(f, g) &= \sup_{x \in X} d(f(x), g(x)) \\ &\leq \sup_{x \in X} (d(f(x), h(x)) + d(h(x), g(x))) \\ &\leq \sup_{x \in X} d(f(x), h(x)) + \sup_{x \in X} d(h(x), g(x)) \\ &= d_{sup}^d(f, h) + d_{sup}^d(h, g) \end{aligned}$$

So d_{sup}^d is a metric function on the non-empty $\mathcal{B}(X, Y)$ and $(\mathcal{B}(X, Y), d_{sup}^d)$ is a metric space.

2. Consider an arbitrary d_{sup}^d -Cauchy sequence on $\mathcal{B}(X, Y)$, $(f_n)_{n \in \mathbb{N}} : f_n \in \mathcal{B}(X, Y) \forall n \in \mathbb{N}$. Then $\forall \varepsilon > 0 \exists n(\varepsilon)$ such that

$$\begin{aligned} d_{sup}^d(f_n, f_m) &< \varepsilon, \forall n, m > n(\varepsilon) \\ \sup_{x \in X} d(f_n(x), f_m(x)) &< \varepsilon, \forall n, m > n(\varepsilon) \\ \forall x \in X, d(f_n(x), f_m(x)) &< \varepsilon, \forall n, m > n(\varepsilon) \end{aligned}$$

so that $(f_n(x))_{n \in \mathbb{N}}$ is a d -Cauchy sequence on Y , $\forall x \in X$.

Here is a breakdown to facilitate your intuition. From one d_{sup}^d -Cauchy sequence in the set of bounded Y -valued functions, $\mathcal{B}(X, Y)$, we can get multiple sequences, $(y_n)_{n \in \mathbb{N}}$, in Y that are d -Cauchy. For each such starting sequence, $(f_n)_{n \in \mathbb{N}}$, there are as many such $(y_n)_{n \in \mathbb{N}}$ sequences as elements in the domain of those f_n , X , and their n -th element is given by $y_n = f_n(x)$.

Furthermore, because (Y, d) is complete, $(f_n(x))_{n \in \mathbb{N}}$ converges to a d -limit in Y , say $\phi_x \in Y$, for all $x \in X$.

So starting from a d_{sup}^d -Cauchy sequence on $\mathcal{B}(X, Y)$ we can generate a collection of d -convergent sequences on Y (one sequence for each $x \in X$). That is, we have that

$$\begin{aligned} &\forall (f_n)_{n \in \mathbb{N}} : (f_n)_{n \in \mathbb{N}} \text{ } d_{sup}^d\text{-Cauchy on } \mathcal{B}(X, Y) \\ &\exists \{(y_n)_{n \in \mathbb{N}} : y_n = f_n(x) \forall n \in \mathbb{N}, (y_n)_{n \in \mathbb{N}} \text{ } d\text{-convergent on } Y, \forall x \in X\} \end{aligned}$$

We can define a function, $f : X \rightarrow Y$, for any such starting sequence and subsequent collection such that $f(x) := \phi_x, \forall x \in X$. To show that $(\mathcal{B}(X, Y), d_{sup}^d)$ is complete, it suffices to show that $d_{sup}^d - \lim f_n = f$ and that $f \in \mathcal{B}(X, Y)$.

Firstly, notice that

$$\begin{aligned} d_{sup}^d(f_n, f) &= \sup_{x \in X} d(f_n(x), f(x)) \\ &= \sup_{x \in X} d(f_n(x), d - \lim_{m \rightarrow +\infty} f_m(x)) \\ &= \sup_{x \in X} \lim_{m \rightarrow +\infty} \left(d(f_n(x), f_m(x)) \right) \end{aligned}$$

This last equality follows by the fact that metrics are continuous functions (*needs proof*).

Now consider $d(f_n(x), f_m(x))$ for some $x \in X$ (notice that it is a real number). Choose an arbitrary fixed $n \in \mathbb{N}$ and think of the real valued sequence $(z_m)_{m \in \mathbb{N}} \in \mathbb{R}$, such that $z_m = d(f_n(x), f_m(x))$ for any given $n \in \mathbb{N}$ and $x \in X$.

$$\begin{aligned} d_{sup}^d(f_n, f) &= \sup_{x \in X} \lim_{m \rightarrow +\infty} z_m \\ &\leq \sup_{x \in X} \sup_{m \geq n} z_m \end{aligned}$$

This holds because we are considering more z_m than just those that “tend to infinity”. Keep in mind that, if we denote z as the limit here, even if $z_m < z, \forall m \in \mathbb{N}$ it still holds that $\sup_{m \in \mathbb{N}} z_m = z$. This logic can be generalized to cases where a finite number of z_m are greater than the limit (if they were infinite, then it wouldn't be a limit).

So we have that

$$\begin{aligned} d_{sup}^d(f_n, f) &\leq \sup_{x \in X} \sup_{m \geq n} d(f_n(x), f_m(x)) \\ &\leq \sup_{x \in X} \sup_{m \geq n} \sup_{x \in X} d(f_n(x), f_m(x)) \\ &= \sup_{m \geq n} \sup_{x \in X} d(f_n(x), f_m(x)) \\ &= \sup_{m \geq n} d_{sup}^d(f_n, f_m) \end{aligned}$$

So we showed that $d_{sup}^d(f_n, f) \leq \sup_{m \geq n} d_{sup}^d(f_n, f_m)$. Now consider the following

$$\begin{aligned} (f_n)_{n \in \mathbb{N}} &\text{ is } d_{sup}^d\text{-Cauchy} \\ \forall \varepsilon > 0, \exists n(\varepsilon) : d_{sup}^d(f_n, f_m) &< \varepsilon & \forall n, m \geq n(\varepsilon) \\ \forall \varepsilon > 0, \exists n(\varepsilon) : \sup_{m \geq n} d_{sup}^d(f_n, f_m) &< \varepsilon & \forall n \geq n(\varepsilon) \\ \forall \varepsilon > 0, \exists n(\varepsilon) : d_{sup}^d(f_n, f) &< \varepsilon & \forall n \geq n(\varepsilon) \end{aligned}$$

thus as $n \rightarrow +\infty, f_n \rightarrow f$ with respect to d_{sup}^d .

Secondly, for any two points $f(x), f(y) \in Y$ and some $n \in \mathbb{N}$ we have that

$$\begin{aligned} \sup_{x, y \in X} d(f(x), f(y)) &\leq \sup_{x, y \in X} d(f_{n(\varepsilon)}(x), f(x)) + \sup_{x, y \in X} d(f_{n(\varepsilon)}(x), f(y)) \\ &\leq \sup_{x, y \in X} d(f_{n(\varepsilon)}(x), f(x)) + \sup_{x, y \in X} d(f_{n(\varepsilon)}(y), f(y)) + \sup_{x, y \in X} d(f_{n(\varepsilon)}(x), f_{n(\varepsilon)}(y)) \\ &\leq \sup_{x \in X} d(f_{n(\varepsilon)}(x), f(x)) + \sup_{y \in X} d(f_{n(\varepsilon)}(y), f(y)) + \sup_{x, y \in X} d(f_{n(\varepsilon)}(x), f_{n(\varepsilon)}(y)) \\ &= 2 \sup_{x \in X} d(f_{n(\varepsilon)}(x), f(x)) + \sup_{x, y \in X} d(f_{n(\varepsilon)}(x), f_{n(\varepsilon)}(y)) \\ &= 2d_{sup}^d(f_{n(\varepsilon)}, f) + \sup_{x, y \in X} d(f_{n(\varepsilon)}(x), f_{n(\varepsilon)}(y)) \end{aligned}$$

where $n(\varepsilon)$ is such that $d_{sup}^d(f_{n(\varepsilon)}, f) < \varepsilon$. This $n(\varepsilon)$ exists since $f_n \rightarrow f$ with respect to d_{sup}^d . Thus the first

additive term is bounded by 2ε . The second additive term is the maximum distance between all values of $f_{n(\varepsilon)}$ on Y . Since $f_{n(\varepsilon)} \in \mathcal{B}(X, Y)$ this number is also bounded. Thus, $\sup_{x, y \in X} d(f(x), f(y)) < +\infty$, which establishes that $f \in \mathcal{B}(X, Y)$, i.e. f is a bounded Y -valued function.

So for (Y, d) complete space, every d_{sup}^d -Cauchy sequence on $\mathcal{B}(X, Y)$ is d_{sup}^d -convergent in $\mathcal{B}(X, Y)$.

Thus, if (Y, d) is a complete metric space, then $(\mathcal{B}(X, Y), d_{sup}^d)$.