MSc Course: Mathematical Analysis - Optional Exercises

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- The following set of exercises is optional.
- The proposed solutions can only be submited at the day of the final exam of June's 2017 exam period. Those should be written in a booklet that contains in its first page your identification data and the number of the written pages. You are to deliver this only at the time that you deliver your exam paper. In the case that you do, you must also make a note in the first page of your exam paper that you are delivering optional exercises. You must make sure that the exam supervisor first inspects the correctness of the information in the first page of the booklet, second signs the first page of the booklet and the note on your exam paper and then encloses the booklet in your exam paper and accepts them.
- The proposed solutions will be graded according to the relevant anouncement.
- The exercises' grade will be valid for any repeater to June's 2017 exam period, exam.
- The instructor retains the right to ask for clarifications on the proposed solutions.

Exercise 1. Given an X-valued sequence $(x_n)_{n\in\mathbb{N}}$, define a $(x_n)_{n\in\mathbb{N}}$- subsequence to be any infinite subset of $(x_n)_{n\in\mathbb{N}}$. Suppose that (X, d) is d - totally bounded and d - complete. Prove that any X-valued sequence $(x_n)_{n\in\mathbb{N}}$ has a d - convergent subsequence. (Such a space is termed (topologically) compact). Prove that the converse also holds.

Exercise 2. Prove that if a metric space is totally bounded then every sequence has a Cauchy subsequence.

Exercise 3. Suppose that (X, d) is compact. Prove that if $f : X \to \mathbb{R}$ is d_I/d - continuous then it is bounded and thereby conclude that $C(X, \mathbb{R}) \subseteq B(X, \mathbb{R})$, where $C(X, \mathbb{R}) = \{f : X \to \mathbb{R},\ f \text{ is } d_I/d\text{- continuous}\}$. Prove that $C(X, \mathbb{R})$ is a closed subset of $\left(B(X, \mathbb{R}), d_{\sup}\right)$. Prove that $C(X, \mathbb{R})$ is $d_{\sup}$- complete.

Exercise 4. Suppose that (X, d) is compact. Prove that if $f : X \to \mathbb{R}$ is d_I/d-continuous then $\arg\max_{x\in X} f(x) \neq \emptyset$. Prove that $\sup : C(X, \mathbb{R}) \to \mathbb{R}$ is $d_I/d_{\sup}$ - continuous. Show that if $f_n, f \in C(X, \mathbb{R})$, $n \in \mathbb{N}$, $\arg\max_{x\in X} f(x) = \{x\}$, then if $d_{\sup}(f_n, f) \to 0$ as $n \to \infty$, and if $x_n \in \arg\max_{x\in X} f_n(x)$, $n \in \mathbb{N}$ then $d(x_n, x) \to 0$ as $n \to \infty$.

Exercise 5. For Y some non empty set and (E, d_E) a complete metric space suppose that $(f_n)_{n\in\mathbb{N}}$ is a $d_{\sup}$-Cauchy sequence inside $B(Y, E)$. Show that

$$\left(\sup_{x,y\in Y} d_E\left(f_n\left(x\right), f_n\left(y\right)\right)\right)_{n\in\mathbb{N}}$$

is a d_u-Cauchy real sequence. Conclude that $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded.

Exercise 6. Consider $\left(B\left(\mathbb{N}, \mathbb{R}\right), d_{\sup}\right)$ and $A = \left\{(x_n)_{n\in\mathbb{N}}, x_n = \begin{cases} 1, & n = i \\ 0, & n \neq i \end{cases}, i \in \mathbb{N}\right\} \subset B\left(\mathbb{N}, \mathbb{R}\right)$. Show that A is $d_{\sup}$ - bounded but not $d_{\sup}$ - totally bounded.

Exercise 7. Given that $(\mathbb{R}, d_I)$ is complete, show that $(\mathbb{R}^n, d_A)$ is complete for any $n > 0$ and A any positive definite $n \times n$ matrix.

Exercise 8. Suppose that (X, d) is compact. Show that for $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, if the sequence (f_n) is d_I/d - equi - Lipschitz then it is $d_{\sup}$ - bounded. Prove that if furthermore $f_n(x) \to f(x)$ as $n \to \infty$, for all $x \in X$, for some $f : X \to \mathbb{R}$, then also $d_{\sup}(f_n, f) \to 0$ as $n \to \infty$, and conclude that the limit f is d_I/d - Lipschitz.

Exercise 9. Prove the Matkowski Fixed Point Theorem:

Theorem. *Suppose that (X, d) is complete, $f : X \to X$, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that:*

1. *g is non-decreasing,*
2. *g is continuous at zero,*
3. *$g(t) = 0$ iff $t = 0$,*
4. *$\lim_{m\to\infty} g^{(m)}(t) = 0, \forall t \in \mathbb{R}_+$, and*
5. *$\forall t > 0, \lim_{m\to\infty} \frac{g^{(m+1)}(t)}{g^{(m)}(t)} = c_t < 1.$*

Then if $\forall x, y \in X$, $d(f(x), f(y)) \leq g(d(x, y))$, f has a unique fixed point, say $x^\star = \lim_{m\to\infty} f^{(m)}(x)$, for all $x \in X$.

Show that this is a generalization of BFPT.

Exercise 10. (Fredholm Integral Equation of the second kind.) Consider $X = C([a, b], \mathbb{R})$ with $d = d_{\sup}$. Suppose that $\omega : [a, b]\times[a, b] \to \mathbb{R}$ is continuous, that $0 < M_\omega \sup_{x,y\in[a,b]} \omega(x, y)$, $h \in X$ and let $\lambda > 0$. Consider the integral equation

$$f(x) = h(x) + \lambda \int_a^b \omega(x, y) f(y)\, dy,\ \forall x \in [a, b]. \tag{1}$$

Show that there exists a unique $f \in X$ that satisfies (1) if $\lambda < \frac{1}{M_{\omega(b-a)}}$.

Exercise 11. (Perron-Frobenius) Remember that $A = \left(a_{i,j}\right)_{i=1,\ldots,q,j=1,\ldots,p}$ with $a_{i,j} \in \mathbb{R}$, $\forall i, j$, is called positive ($A > 0$) iff $a_{i,j} > 0,\ \forall i, j$. Show that if $p = q$ and $A > 0$ then A has at least one positive eigenvalue and at least one positive eigenvector.