MSc Course: Mathematical Analysis - Optional Exercises (2020-21)

Stelios Arvanitis

- The following set of exercises is optional.
- The details of the (electronic) submission process will be announced in time.
- The instructor retains the right to ask for clarifications on the proposed solutions.

Exercise 1. Given an X-valued sequence $(x_n)_{n\in\mathbb{N}}$, define a $(x_n)_{n\in\mathbb{N}}$- subsequence to be any infinite subset of $(x_n)_{n\in\mathbb{N}}$. Prove that if a metric space is totally bounded then every sequence has a Cauchy subsequence.

Exercise 2. Consider $\left(B(\mathbb{N},\mathbb{R}), d_{\text{sup}}\right)$ and $A = \left\{ (x_n)_{n\in\mathbb{N}},\ x_n = \begin{cases} 1, & n = i \\ 2, & n \neq i \end{cases},\ i \in \mathbb{N} \right\} \subset B(\mathbb{N},\mathbb{R})$. Show that A is d_{sup} - bounded but not d_{sup} - totally bounded.

Exercise 3. Suppose that (X,d) is compact. Prove that if $f : X \to \mathbb{R}$ is d_I/d - continuous then it is bounded and thereby conclude that $C(X,\mathbb{R}) \subseteq B(X,\mathbb{R})$, where $C(X,\mathbb{R}) = \{f : X \to \mathbb{R},\ f \text{ is } d_I/d\text{- continuous}\}$. Prove that $C(X,\mathbb{R})$ is a closed subset of $\left(B(X,\mathbb{R}), d_{\text{sup}}\right)$. Prove that $C(X,\mathbb{R})$ is d_{sup}- complete.

Exercise 4. Suppose that (X,d) is compact. Show that for $f_n : X \to \mathbb{R},\ n \in \mathbb{N}$, if the sequence (f_n) is d_u/d - equi - Lipschitz then it is d_{sup} - bounded. Prove that if furthermore $f_n(x) \to f(x)$ as $n \to \infty$, for all $x \in X$, for some $f : X \to \mathbb{R}$, then also $d_{\text{sup}}(f_n, f) \to 0$ as $n \to \infty$, and conclude that the limit f is d_u/d - Lipschitz.

Exercise 5. Given that $(\mathbb{R}, d_I)$ is complete, show that $(\mathbb{R}^n, d_A)$ is complete for any $n > 0$ and A any positive definite $n \times n$ matrix.

Exercise 6. Suppose that (X,d) is compact. Show that for $f_n : X \to \mathbb{R},\ n \in \mathbb{N}$, if the sequence (f_n) is d_I/d - equi - Lipschitz then it is d_{sup} - bounded. Prove that if furthermore $f_n(x) \to f(x)$ as $n \to \infty$, for all $x \in X$, for some $f : X \to \mathbb{R}$, then also $d_{\text{sup}}(f_n, f) \to 0$ as $n \to \infty$, and conclude that the limit f is d_I/d - Lipschitz.

Exercise 7. Prove the Matkowski Fixed Point Theorem:

Theorem. *Suppose that (X,d) is complete, $f : X \to X$, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that:*

1. *g is non-decreasing,*
2. *g is continuous at zero,*

3. $g(t) = 0$ *iff* $t = 0$,

4. $\lim_{m\to\infty} g^{(m)}(t) = 0$, $\forall t \in \mathbb{R}_+$, *and*

5. $\forall t > 0$, $\lim_{m\to\infty} \frac{g^{(m+1)}(t)}{g^{(m)}(t)} = c_t < 1$.

Then if $\forall x, y \in X$, $d(f(x), f(y)) \leq g(d(x,y))$, f *has a unique fixed point, say* $x^\star = \lim_{m\to\infty} f^{(m)}(x)$, *for all* $x \in X$.

Show that this is a generalization of BFPT.

Exercise 8. (Fredholm Integral Equation of the second kind.) Consider $X = C([a,b], \mathbb{R})$ with $d = d_{\mathsf{sup}}$. Suppose that $\omega : [a,b] \times [a,b] \to \mathbb{R}$ is continuous, that $\omega(x,y) \geq 0,\ \forall x, y \in [a,b]$, that $0 < M_\omega := \sup_{x,y\in[a,b]} \omega(x,y)$, $h \in X$ and let $\lambda > 0$. Consider the integral equation

$$f(x) = h(x) + \lambda \int_a^b \omega(x,y)\, f(y)\, dy,\ \forall x \in [a,b]. \tag{1}$$

Show that there exists a unique $f \in X$ that satisfies (1) if $\lambda < \frac{1}{M_\omega (b-a)}$.

Exercise 9. (Perron-Frobenius) Remember that $A = \left(a_{i,j}\right)_{i=1,\ldots,q, j=1,\ldots,p}$ with $a_{i,j} \in \mathbb{R},\ \forall i, j$, is called positive ($A > 0$) iff $a_{i,j} > 0,\ \forall i, j$. Show that if $p = q$ and $A > 0$ then A has at least one positive eigenvalue and at least one positive eigenvector. (Hint: study and use the Brouwer FPT)