Metric Entropy and the Sample Paths of the Wiener Process

We apply the Dudley's entropy integral theorem to the following example of Gaussian process.

[With correction in green - 27/05/2017]

Definition [Wiener Process] For $\Theta = [0,1]$ the process that satisfies the following properties: $(W : \Omega \to \mathbb{R}^{[0,1]})$

1. $W(\omega, 0) = 0$, $\mathbb{P}$ a.s. (thereafter we supress dependence on Ω)
2. for any $t, s \in [0,1]$ and $h \in \mathbb{R}$: $t+h, s+h \in [0,1]$ the increment $W(t) - W(s)$ is independent of $W(t+h) - W(s+h)$, and $W(t) - W(s)$, $W(t+h) - W(s+h)$ are identically distributed.
3. $W(t) - W(s) \sim N(0, |t-s|)$, for any $t, s \in [0,1]$,

is called a standard Wiener process on $[0,1]$. $\square$

It can be proven that:

a. W exists (via the Daniell-Kolmogorov Theorem and possibly via what is termed as Karhunen-Loeve Representation),

b. W is Gaussian, and $W(t) \sim N(0,t)$ (the second part follows trivially from that $W(t) = W(t) - W(0)$, $\mathbb{P}$ a.s.).

Consider $d^* : [0,1] \times [0,1] \to \mathbb{R}$, defined by, $d^*(t,s) = \left[\mathbb{E}\left[W(t) - W(s)\right]^2\right]^{1/2} = \sqrt{|t-s|}$, and we already now that d^* is a metric on $[0,1]$, (we know that it is a pseudo-metric from Dudley's Theorem - why it is actually a metric?). Furthermore we have that $\forall x \in [0,1]$, $\forall \varepsilon > 0$, $O_d(x, \varepsilon) \le O_{d^*}(x, \sqrt{\varepsilon}) \Rightarrow O_d(x, \varepsilon^2) \le O_{d^*}(x, \varepsilon)$. This directly implies (why?) that $N(\varepsilon, d^*, [0,1]) \le N(\varepsilon^2, d, [0,1]) \sim c_*/\varepsilon^2$ as $\varepsilon \to 0^+$ for some $c_* > 0$, which then implies that $[0,1]$ is d^*-totally bounded (why?). By choosing $\delta > 0$, small enough, the above imply that for constants $C > 0$, (C may change its value below)

$$\int_0^\delta \left[\ln N(\varepsilon, d^*, [0,1])\right]^{1/2} d\varepsilon \le \int_0^\delta \left[\ln N(\varepsilon^2, d, [0,1])\right]^{1/2} d\varepsilon$$

$$\le C \int_0^\delta \sqrt{\ln[c_*] - 2\ln\varepsilon}\, d\varepsilon \le 2c \int_0^\delta \varepsilon' \sqrt{|\ln\varepsilon|}\, d\varepsilon = 2c\left[\varepsilon\sqrt{|\ln\varepsilon|}\,\Big|_0^\delta\right] - 2c \int_0^\delta \varepsilon \frac{1}{2\sqrt{|\ln\varepsilon|}} \frac{1}{\varepsilon}\, d\varepsilon$$

$$\le 2c\,\delta\sqrt{|\ln\delta|} - 2c\frac{1}{\sqrt{|\ln\delta|}} \int_0^\delta d\varepsilon = 2c\left[\delta\sqrt{|\ln\delta|} - \frac{1}{\sqrt{|\ln\delta|}}\right] < +\infty.$$

Hence the aforementioned theorem implies that

$W(\omega,\cdot):[0,1]\to\mathbb{R}$ is d_u/d^* continuous, for P almost all $\omega\in\Omega$,

which then implies that (why?)

$W(\omega,\cdot):[0,1]\to\mathbb{R}$ is d_u/d_u continuous, for P almost all $\omega\in\Omega$. $\square$