An Example of a Uniformly Bounded Set

Let $Y = [0,1]$ and consider $X = B([0,1], \mathbb{R})$ equipped with d_{sup}. Furthermore consider for some $L>0$, $A_L := \{ f : [0,1] \to \mathbb{R} : f(0) = 0,\ \forall x,y \in [0,1],\ |f(x)-f(y)| \leq L|x-y| \}$. Notice that:

1. $A \neq \emptyset$, since $f(x) = Lx : [0,1] \to \mathbb{R} \in A_L$ or
$g(x) := L(e^{-x}-1) : [0,1] \to \mathbb{R} \in A_L$ since
$\forall x,y \in [0,1]$, due to the mean value theorem (potentially applied w.r.t. one sided derivatives),
$$g(x) - g(y) = -L e^{-x^*}(x-y) \quad \text{for some } x^* \text{ between}$$
x and y, hence,
$$|g(x)-g(y)| \leq L \sup_{z \in [0,1]} |e^{-z}|\, |x-y| = L|x-y|.$$

2. $A \subseteq B([0,1], \mathbb{R})$, since if $f \in A$ then
$$\sup_{x\in[0,1]} |f(x)| = \sup_{x \in [0,1]} |f(x) - f(0)| \leq \sup_{x\in[0,1]} L|x| = L < +\infty.$$

3. A is d_{sup}-bounded since, similarly to the above
$$\sup_{f\in A} \sup_{x \in [0,1]} |f(x)| = \sup_{f \in A} \sup_{x\in[0,1]} |f(x)-f(0)| \leq \sup_{f\in A} \sup_{x \in [0,1]} L|x|$$
$$= L < +\infty. \ \square$$

As we will see later on during the course, the L-condition that the members of A satisfy implies that A is a set of equi-Lipschitz functions.

Exercise. Consider $B_{L,c} := \{ f : [0,1] \to \mathbb{R} : |f(0)| \leq c,\ \forall x,y \in [0,1],\ |f(x)-f(y)| \leq L|x-y| \}$ for some L as above and $c > 0$. Show that $B_{L,c}$ is d_{sup}-bounded.