Consider an arbitrary metric space (X,d). The fact that the metric enables the consideration of a notion of distance between pairs of elements of the carrier set X, implies that we can attribute to each element of the space, a "piece" of X constructed from every element with distance less than (or equal) to some prescribed number from the aforementioned one.

Definition. For $x \in X$, $\varepsilon > 0$, the open ball with center x and of radius ε, is defined as $O_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$.

Analogously the closed ball with center x and of radius ε, is defined as $O_d[x,\varepsilon] = \{y \in X : d(x,y) \leq \varepsilon\}$. $\square$

Lemma 1. For any $x \in X$, $\varepsilon > 0$, $O_d(x,\varepsilon) \subseteq O_d[x,\varepsilon]$.

Proof. $x \in O_d(x,\varepsilon)$ since $d(x,x) \overset{ii}{=} 0 \Rightarrow d(x,x) < \varepsilon \ \forall \varepsilon > 0$. $\square$

Lemma 2. For any $x \in X$, $\varepsilon, \delta > 0$ with $\varepsilon < \delta$

$$O_d(x,\varepsilon) \subseteq O_d[x,\varepsilon] \subseteq O_d(x,\delta) \subseteq O_d[x,\delta]$$

Proof. Due to Lemma 1 $\exists\, y \in O_d(x,\varepsilon) \Rightarrow d(x,y) < \varepsilon \Rightarrow$

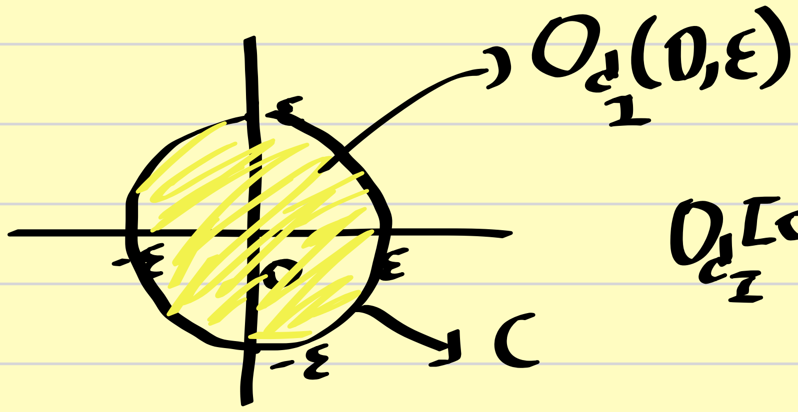
$$d(x,y) \leq \varepsilon \overset{\varepsilon < \delta}{\Rightarrow} d(x,y) < \delta \Rightarrow d(x,y) \leq \delta. \ \square$$

Hence open and closed balls are never empty, and for x fixed they are monotone functions of the radius w.r.t. set inclusion.

Examples

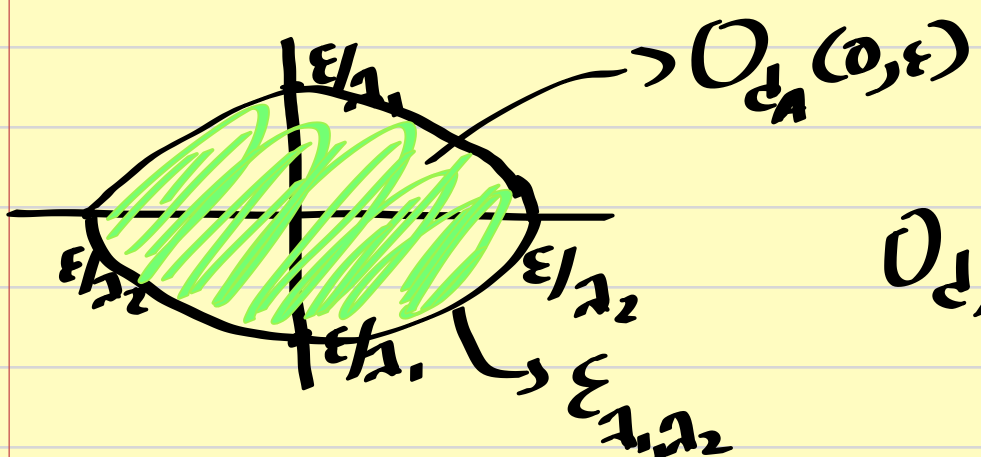
1. $(\mathbb{R}, d_u)$, $O_{d_u}(x,\varepsilon) = (x-\varepsilon, x+\varepsilon)$, $O_d[x,\varepsilon] = [x-\varepsilon, x+\varepsilon]$. $\square$

2. $(\mathbb{R}^2, d_I)$



$O_{d_I}[0,\varepsilon] = O_{d_I}(0,\varepsilon) \cup C$ $\square$

3. $(\mathbb{R}^2, d_A)$

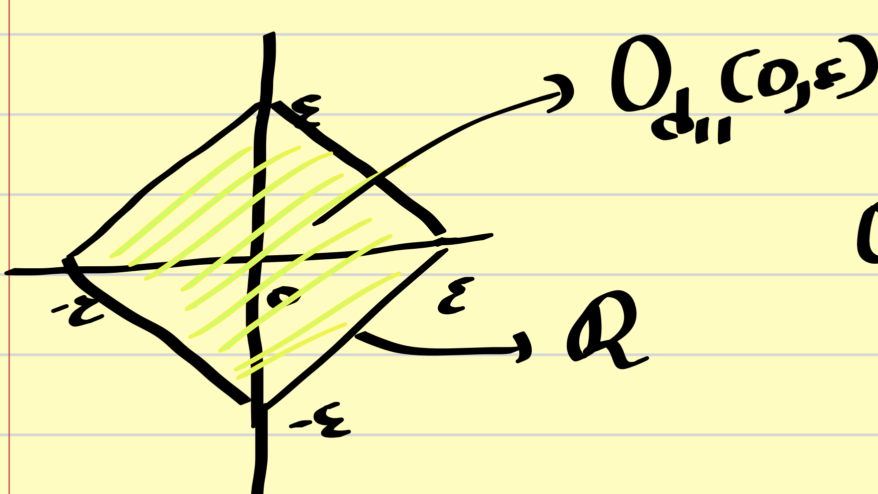


$O_{d_A}[0,\varepsilon] = O_{d_A}(0,\varepsilon) \cup \mathcal{E}_{\lambda_1,\lambda_2}$

where λ_1 and λ_2 are the eigenvalues of A in the relevant order. The open ball are the regions bounded by the ellipsoid defined by A, $\mathcal{E}_{\lambda_1,\lambda_2}$, with maximum radius equal to $\max\{\lambda_1,\lambda_2\}$, minimum radius equal to $\min\{\lambda_1,\lambda_2\}$, principal axis the ones defined by the eigenvectors (in the depiction the eigenvector to λ_1 is $(0,1)$ and to λ_2 is $(1,0)$). When $\lambda_1=\lambda_2$ we obtain a sphere (hypersphere in higher dimensions). The closed balls are constructed as the unions of the aforementioned region with the ellipsoid. When $A = \mathrm{Id}_{2\times 2}$ we obtain the previous example. $\square$

Remark. The previous two examples hint that different metrics may endow X with different "geometrical" properties.

4. $(\mathbb{R}^2, d_{11})$



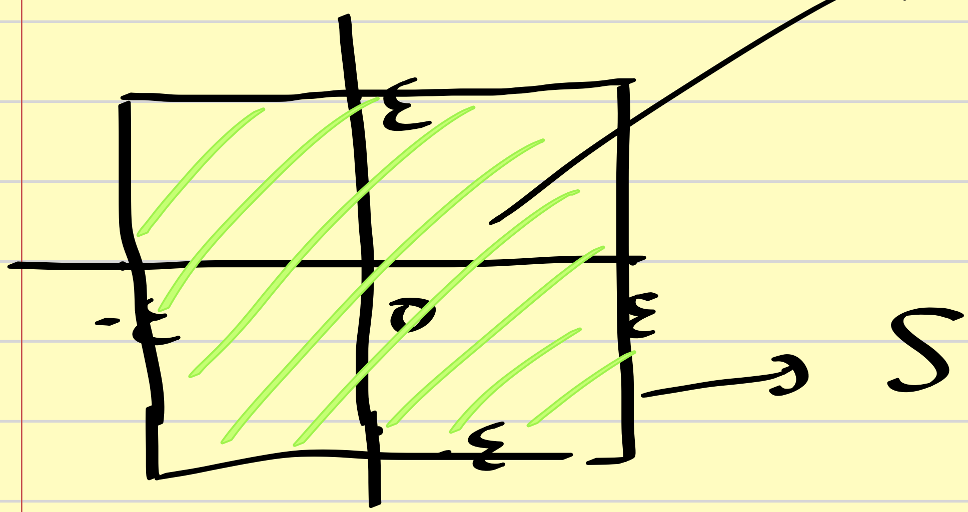
$$O_{d_{11}}[0,\varepsilon] = O_d(0,\varepsilon) \cup R$$

$\square$

Examine examples 2-4 when $x\neq 0$ and/or $k\neq 2$.

5. $(\mathbb{R}^3, d_{max})$

$\to O_{d_{max}}(0,\varepsilon)$



$$O_{d_{max}}[0,\varepsilon] = O_{d_{max}}(0,\varepsilon) \cup S$$ □

6. $(\mathbb{R}, d_e)$

$O_{d_e}(x,\varepsilon) = \{y \in \mathbb{R} : |e^y - e^x| < \varepsilon\}$, but $|e^y - e^x| < \varepsilon$

$\iff -\varepsilon < e^y - e^x < \varepsilon \iff e^x - \varepsilon < e^y < e^x + \varepsilon \iff$

$y \in (k, \ln(e^x+\varepsilon))$ where $k = \begin{cases} -\infty & \text{if } e^x \leq \varepsilon \\ \ln(e^x - \varepsilon) & \text{if } e^x > \varepsilon. \end{cases}$

Hence e.g. the intervals $(-\infty, u)$, $u > 0$, are considered of finite radius w.r.t. d_e, something that reinforces the statement in the previous remark! The closed balls are of the form $[k, \ln(e^x+\varepsilon)]$ when $k > -\infty$ or $(-\infty, \ln(e^x+\varepsilon)]$ when $k = -\infty$. □

7. X arbitrary, $d = d_\delta$

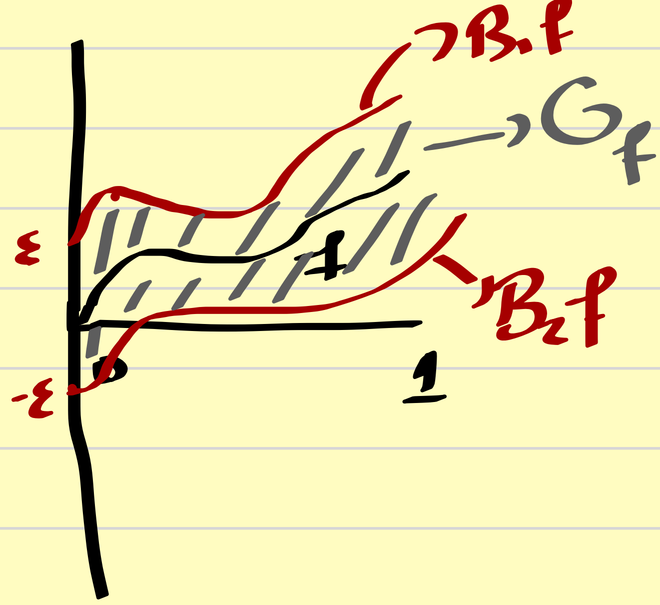
Notice that $O_{d_\delta}(x,\varepsilon) = \begin{cases} X & \varepsilon > 1 \\ \{x\} & \varepsilon \leq 1 \end{cases}$, and

$$O_{d_\delta}[x,\varepsilon] = \begin{cases} X & \varepsilon \geq 1 \\ \{x\} & \varepsilon < 1 \end{cases}.$$

Notice that when $\varepsilon > 1$ or $\varepsilon < 1$ $O_d(x,\varepsilon) = O_d[x,\varepsilon]$ and when $1 < \varepsilon < \varepsilon'$ or $\varepsilon < \varepsilon' < 1$

$O_{d_\delta}(x,\varepsilon) = O_{d_\delta}[x,\varepsilon] = O_{d_\delta}(x,\varepsilon') = O_{d_\delta}[x,\varepsilon']$ implying that the set theoretic inclusions in Lemma 1 might hold as equalities along with that discrete spaces might have "peculiar" properties. $\square$

8. $Y = [0,1]$, $X = B(Y,\mathbb{R})$, $d = d_{sup}$



Beware: $O_{d_{sup}}(f,\varepsilon) = \{ g \in B(Y,\mathbb{R}) : \text{graph}(g) \subseteq G_f \}$

and

$$O_{d_{sup}}[f,\varepsilon] = \{ g \in B(Y,\mathbb{R}) : \text{graph}(g) \subseteq G_f \cup B_1 f \cup B_2 f \}$$ $\square$

The two following results, a separation, and a countability property imply (as we will later see) important properties for metric spaces.

Lemma 2. [separation by open balls]. For any $x,y\in X: x\neq y$, $\exists\ \varepsilon_1,\varepsilon_2>0$ such that $O_d(x,\varepsilon_1)\cap O_d(x,\varepsilon_2)=\phi$.

Proof. $x\neq y \iff d(x,y)=\varepsilon>0$. Set $\varepsilon_1=\varepsilon_2=\varepsilon/2$.
We claim that $O_d(x,\varepsilon/2)\cap O_d(y,\varepsilon/2)=\phi$.
Suppose not. Then $\exists\ z\in O_d(x,\varepsilon/2)\cap O_d(y,\varepsilon/2)$.
Hence $d(x,z)<\varepsilon/2$ and $d(z,y)<\varepsilon/2$ (due i).
Now $\varepsilon=d(x,y)\underset{iv}{\leq} d(x,z)+d(z,y)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.
Hence the balls must be disjoint. $\square$

Thereby any two distinct elements can be separated by open (or closed – prove it) balls.

This has important implications in the topology of metric spaces, e.g. the uniqueness of limits.

Lemma 3. For any $x\in X$ there exists a countable set of open balls, say Δ_x, such that for any $O_d(x,\varepsilon)$, $\exists$ $A\in\Delta_x: A\subseteq O_d(x,\varepsilon)$ [First Countability of Metric Spaces]

Proof. For $x\in X$, choose $\Delta_x:=\{O_d(x,1/n), n\in\mathbb{N}\}$ which is countable (why?). For any $\varepsilon>0$, $\exists\, n(\varepsilon)\in\mathbb{N}$: $\frac{1}{n(\varepsilon)}<\varepsilon$ (eg $n(\varepsilon)$ = smallest natural greater than ε). Then $O_d(x,1/n)\subseteq O_d(x,\varepsilon)$ since $y\in O_d(x,1/n)\Leftrightarrow d(x,y)<1/n\Rightarrow d(x,y)<\varepsilon\Rightarrow y\in O_d(x,\varepsilon)$. $\square$

This property implies that the consideration of the issue of convergence in metric spaces can be performed using sequences and not more complex objects, such as nets.

Exercises

1. Show that open and closed balls can be defined for pseudo-metric spaces. $\square$
2. Given 1, what $O_{d_A}(0,1)$ looks like, for $X=\mathbb{R}^2$ and $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$? $\square$
3. Show that Lemma 2 need not be true in pseudo-metric spaces. $\square$
4. Show that Lemma 3 holds for pseudo-metric spaces. $\square$
5. Show that Lemma 3 holds for closed balls in (pseudo-) metric spaces. $\square$

[The notes are in a state of perpetual correction. They do not substitute the lectures. Please report any typos to stelios@aueb.gr or the course's e-class.]