Non Convex Balls.

The following is an example of a metric on $\mathbb{R}$ with non-convex open (closed) balls. Suppose that $X = \mathbb{R}$. Define $d^*: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ via d_u as follows:

$$\text{if } x,y \in \mathbb{R},\quad d^*(x,y) := \begin{cases} 0 & x=y \\ 2 & |x-y| \in (k,k+1], \text{ any } k \text{ even} \\ 1 & |x-y| \in (k,k+1], \text{ any } k \text{ odd} \end{cases}.$$

d^* is a well defined metric (show it!). Notice that for any $x \in \mathbb{R}$ we have that

$$O_{d^*}(x,\varepsilon) = \begin{cases} \{x\}, & \varepsilon \le 1 \\ \{x\} \cup \Big(\bigcup_{k \text{ odd}} [-k-1+x, x-k)\Big) \cup \Big(\bigcup_{k \text{ odd}} (k+x, k+1+x]\Big), & 1 < \varepsilon \le 2 \\ \{x\} \cup \Big(\bigcup_{k \text{ even}} [-k-1+x, x-k)\Big) \cup \Big(\bigcup_{k \text{ even}} (k+x, k+1+x]\Big), & 2 < \varepsilon \end{cases}.$$

Find the closed balls. Notice that when $\varepsilon > \varepsilon$, $O_{d^*}(x,\varepsilon)$ is a non-convex subset of $\mathbb{R}$ (convexity is essentially an algebraic property, hence does not depend on d).

Out of the scope of the course: A subset of a metric space (X,d) is called d-connected iff it cannot be expressed as a disjoint union of d-open sets. Otherwise it is called d-disconnected (this is weaker than path connectedness). Notice that iff $\varepsilon > 1$ $O_{d^*}(x,\varepsilon)$ is d_u-disconnected, it is however d^*-connected since as we will be able to see no interval of the form $(y, y+1]$ is d^* open. Hence such notions require attention as they generally depend on the metric (compare it with convexity). However it is easy to construct an example of a metric space (X,d) with (some) d-disconnected open balls. E.g. suppose that $\#X \ge 2$, and consider (X, d_s). We have that

$$O_{d_s}(x,2) = X = \bigcup_{y\in X} \{y\} = \bigcup_{y \in X} O_{d_s}(y,1).$$

The previous can be easily generalized. For (X,d) an arbitrary metric space define $d^*: X\times X \to \mathbb{R}$ by

$$\text{if } x,y \in X,\quad d^*(x,y) := \begin{cases} 0, & x=y \\ 2, & d(x,y) \in (k,k+1], \ k \text{ even} \\ 1, & d(x,y) \in (k,k+1], \ k \text{ odd} \end{cases},$$

for which we have that if $x \in X$, $\varepsilon > 0$

$$O_{d^*}(x,\varepsilon) = \begin{cases} \{x\}, & \varepsilon \leq 1 \\ \{x\} \cup \left(\bigcup_{k \text{ odd}} \left(O_d[x,k+1] - O_d[x,k] \right) \right), & 1 < \varepsilon \leq 2 \\ \{x\} \cup \left(\bigcup_{k \text{ even}} \left(O_d[x,k+1] - O_d[x,k] \right) \right), & 2 < \varepsilon \end{cases}.$$

Exercise. Show that d^* is a metric, that the open balls have the aforementioned form and find the closed balls (notice that if $A, B \subseteq X$, $A - B = \{x \in A, x \notin B\}$).

Exercise. Generalize d^* in such a way so that d_s can be obtained as a subcase from the generalization.