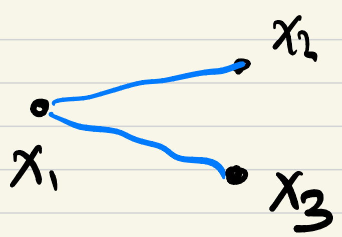
An Example of a Metric based on Graph Structure

Suppose that $V \neq \emptyset$ (set of vertices) and finite

Definition. An unordered pair of elements of V is a set $\{x_i, x_j\}$ with $x_i, x_j \in V$. ($\{x_i, x_j\}$ is not actually a vector of elements of V, since $\{x_i, x_j\} = \{x_j, x_i\}$, and this justifies the term "unordered")

Definition. An unordered finite graph on V is a pair $G = (V, \mathcal{E})$ where $\mathcal{E}$ is a collection of unordered pairs of elements of V. [$\mathcal{E}$ is the set of edges in the Graph]

E.g. $V = \{x_1, x_2, x_3\}$ $\quad \mathcal{E} = \{\{x_1, x_2\}, \{x_1, x_3\}\}$



• vertices $\simeq$ trivial pairs $\{x_i, x_i\}$

— elements of $\mathcal{E}$

can be considered as representing symmetric relations between the vertices

Remark. Given the identification $\{x_i, x_j\} = \{x_j, x_i\}$, at most one edge can exist between any pair of vertices. Such graphs are called *simple*. Since G is simple and

V is finite, $\mathcal{E}$ must be also finite (why?). Such graphs (V finite and $\mathcal{E}$ finite) are called finite.

Definition. If $x,y \in V$ and $\{x,y\} \in \mathcal{E}$ then x,y are called adjunce (connected by an edge $x \text{ adj } y$). A path between x,y ($P_{x,y}$) is a finite sequence in V

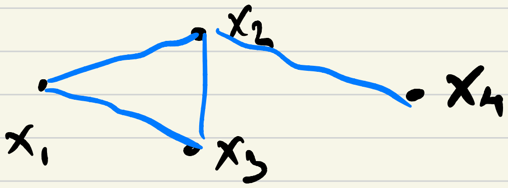
↳ essentially a vector, why?

$(x_1, x_2, \dots, x_i, x_{i+1}, \dots x_n)$, with $x_1 = x$, $x_n = y$ and $x_i \text{ adj } x_{i+1}$, $\forall i = 1, \dots, n-1$.

Remark. A path is called a loop iff $x=y$. A loop is called trivial iff it is a singleton loop (why do trivial loops exist, by the above?)

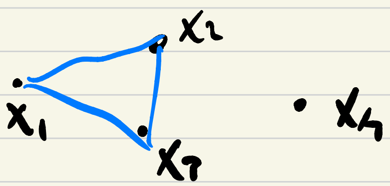
Definition. A graph G is called connected, iff $\forall x,y \in V$ there exists a path $P_{x,y}$ (every vertex is reachable by any vertex via some path)

E.g. $V = \{x_1, x_2, x_3, x_4\}$ $\mathcal{E} = \{\{x_1,x_2\}, \{x_1,x_3\}, \{x_2,x_3\}, \{x_2,x_4\}\}$



$G = (V, \mathcal{E})$ connected (why?)

$\mathcal{E}^* = \{\{x_1,x_2\}, \{x_1,x_3\}, \{x_2,x_3\}\}$



$G^* = (V, \mathcal{E}^*)$

Not-connected (why?)

- Some Algebra on Paths

- Transposition: if $P_{x,y} = (x, x_2, \dots, x_i, x_{i+1}, \dots, x_{n-1}, y)$
then $P'_{x,y} = (y, x_{n-1}, \dots, x_{i+1}, x_i, \dots, x_2, x)$

Exe: $P'_{x,y} = P_{x,y}$ iff $P_{x,y}$ is a trivial loop.

- Concatenation: if $P_{x,y} = \{x, x_2, \dots, x_i, x_{i+1}, \dots, x_{n-1}, y\}$
and $P_{y,z} = \{y, y_2, \dots, y_j, y_{j+1}, \dots, y_{m-1}, z\}$ then

$$P_{x,y} \bullet P_{y,z} := \{x, x_2, \dots, x_i, x_{i+1}, \dots, x_{n-1}, y, y_2, \dots, y_j, y_{j+1}, \dots, y_{m-1}, z\}$$

[Concatenation occurs only on conformal paths,
i.e. $P_{x,y} \bullet P_{w,z}$ is definable iff $y = w$]

Exe: $(P_{x,y} \bullet P_{y,z})' = P'_{y,z} \bullet P'_{x,y}$.

- length of a path

Definition. If $P_{x,y}$ is a path then:

$$\text{length}(P_{x,y}) = \# P_{x,y} - 1 \,.$$

↳ number of elements of the sequence

Exe.
$\text{length}(P_{x,z} \bullet P_{z,y}) = \text{length}(P_{x,z}) + \text{length}(P_{z,y})$

Remark. 1. If G is finite then the function length assumes its values on $\mathbb{N}$ (why?)

2. If G is finite and connected then the optimization $\ell_{xy} := \min\{\text{length}(P_{xy}), P_{xy} \text{ is a path from } x \text{ to } y\}$

is well defined (it always produces a natural number)

since
- G connected $\Leftrightarrow$ collection of paths from $x,y \neq \emptyset$, $\forall x,y \in V$
- G finite $\Leftrightarrow$ " " " finite, $\forall x,y \in V$

(if V was "simply" finite then it could be possible that $\exists x,y \in V$: $l_{x,y} = +\infty$ - why?)

From now on G will be simple, finite, and connected:

3. $\forall x,y \in V$, $l_{x,y} = l_{y,x}$ (why? use transposition)

4. $\forall x,y,z \in V$, $l_{x,y} \leq l_{x,z} + l_{z,y}$ since

$$l_{x,y} \leq \text{length}(P_{x,z} \cdot P_{z,y})$$

has by construction an optimal property

$$\overset{\text{by exe.}}{=} \text{length}(P_{x,z}) + \text{length}(P_{z,y})$$

and the $\leq$ inequality above continuous to hold if $P_{x,z}$ and $P_{z,y}$ in the inequality are chosen optimally. ∎

5. $l_{x,y} = 0$ iff $x = y$ (why?)

We are now ready to construct a metric on V (not on G) based on G.

Metric via G. Let $V \neq \emptyset$, finite, and G a simple connected graph on V

$d_G : V \times V \to \mathbb{N}$, defined by

$x, y \in V$, $d_G(x,y) := l_{x,y}$ is a well defined metric.

- d_G is a well-defined function with values on $\mathbb{N}$ by Rem. 2.
- $d_G(x,y) = 0 \iff x = y$ by Rem 5.
- d_G is symmetric by Rem 3.
- d_G satisfies the triangle inequality by Rem 4.

Remark. If V represents economic actors, and G interconnections between them w.r.t. optimal decisions, (e.g. local games), d_G could reflect the "strategic interdependence" between the actors x, y in terms of their optimal decision making.