[Contains correction in Red – 13/5/17]

Further Fixed Point Theory: Preparation for the Brouwer FPT

We will be examining preparatory notions for the Brouwer FPT. Remember that a topological space, is a pair (X,τ_X) for $X\neq\emptyset$ and τ_X a topology on X. In the present course the topologies we are dealing with are "usually" generated by a metric d_X, as described elsewhere.

Definition. A topological space (X,τ_X) has the fixed point property **(fpp)** iff **every** τ_X/τ_X-continuous self map defined on X has a fixed point. $\square$

[Counter-] Examples.

1. $X=(0,1)$, τ_X generated by d_I. It doesn't have the fpp since, e.g. $f:X\to X$, $f(x)=x^2$, is continuous but the equation $x(1-x)=0$ does not have any solution in X. $\square$
2. X is arbitrary, τ_X generated by d_δ. We have proven that any $f:X\to X$ is d_δ/d_δ-continuous. X has the fpp iff it is a singleton. ($\Rightarrow$) Trivial. ($\Leftarrow$) For $x_1,x_2\in X$, define $f:X\to X$, by
$$f(x)=\begin{cases} x_1, & x\neq x_1\\ x_2, & x=x_1\end{cases}.$$
If $x_1\neq x_2$, f does not have a fixed point. $\square$
3. We will later on prove that $X=[0,1]$, τ_X generated by d_I has the fpp. $\square$

Topological Inheritance of the fpp

Definition. (X,τ_X), (Y,τ_Y) are homeomorphic, iff $\exists\ f:X\to Y$ that is a bijection, and such that f is τ_Y/τ_X-continuous and f^{-1} is τ_X/τ_Y-continuous. Such an f is termed homeomorphism. $\square$

Comment. Homeomorphisms are the morphisms between topological spaces. Vaguely a homeomorphism transforms a topological space without creating tears or holes, etc.

Proposition [Hom] Suppose that (X,τ_X) and (Y,τ_Y) are homeomorphic and that (X,τ_X) has the fpp. Then (Y,τ_Y) has also the fpp. $\square$

Proof. Suppose that $g: Y \to Y$ is continuous and $f: X \to Y$ is a homeomorphism. Then $g^* := f^{-1} \circ g \circ f : X \to X$ is a continuous (as a composition of continuous functions) self map on X. Hence g^* has a fixed point, say x^*. Thereby,

$$x^* = f^{-1}(g(f(x^*))) \iff f(x^*) = f(f^{-1}(g(f(x^*)))) \iff$$

$$f(x^*) = g(f(x^*)),$$ hence $f(x^*) \in Y$ is a fixed point of g. $\square$

Comment. Hence the fpp is a topological invariant.

Retractions – Retracts

Definition. Suppose that $\emptyset \neq S \subseteq X$ for (X,τ_X) a topological space. ($A \in \tau_S \iff A = S \cap A^*$ for some $A^* \in \tau_X$). (S,τ_S) is a retract of (X,d_X) iff $\exists\, r: X \to S$ that is τ_S/τ_X-continuous, such that $r(s) = s$, $\forall s \in S$. $\square$ (r is termed retraction)

We will immediately consider the issue of inheritance of the fpp by a retract, and then examine more thoroughly the notion.

Proposition [Retr] (Borsuk) Suppose that (X,τ_X) has the fpp and S is a retract. Then (S,τ_S) has the fpp. $\square$

Proof. Consider $g: S \to S$ continuous, and $r: X \to S$ a retract. $g^* := g \circ r : X \to S \subseteq X$ is a continuous (as a composition of continuous functions) self map on X. Hence it has a fixed point, say x^*. Thereby,

$$x^* = g^*(x^*) \iff$$
$$x^* = g(r(x^*)) \quad (1).$$

But $g^*: X \to S$ hence $x^* \in S$, but then since r is a retract, $r(x^*) = x^*$ and thereby (1) gives that

$$x^* = g(x^*)$$ and the result follows from that g is arbitrary. $\square$

Examples and Counterexamples of retractions and retracts

(Counter-) Example. Consider $X = [0,1]$ (with the topology obtained from the usual metric) and $S = (0,1)$ (i.e. the interior of X). S cannot be a retract of X. This is due to that if it were, then there would exist a continuous $r: [0,1] \to (0,1)$, such that $r(x) = x$ if $x \in (0,1)$. For $x_n = \frac{1}{n+1}$, the continuity of r implies that $\lim_{n \to +\infty} r(x_n) = r(\lim_{n \to +\infty} x_n) = r(0)$. But $\forall n > 0$, $x_n \in (0,1) \Rightarrow r(x_n) = x_n$

$= \frac{1}{n+1} \to 0$. Hence $r(0) = 0$ which is impossible. $\square$

Example. $X \neq \emptyset$, endowed with the topology generated by the discrete metric. $\emptyset \neq S \subseteq X$. Define $r: X \to S$, by $r(x) = \begin{cases} x, & x \in S \\ y, & x \notin S \end{cases}$ for some $y \in S$.

r is a retraction since every function defined on a discrete space is appropriately continuous. $\square$

(Counter-) Example. $X = [0,1]$, $S = \{0,1\}$, τ_X the one generated by the usual metric. We will see using the Brouwer FPT that S is not a retract of X. $\square$

The following two examples will take the form of lemmata.

Lemma [CC]. Suppose that $X \subseteq \mathbb{R}^n$ and τ_X is generated by d_I. If S is (d_I-) compact and convex, then it is a retraction of X. $\square$

Proof. We will construct a retraction. Define $r : X \to S$ by $r(x) = \underset{y \in S}{\arg\min}\, d_I(x,y)$. i. For any $x \in X$, $d(x,\cdot) : S \to \mathbb{R}$ is continuous and since S is compact $\underset{y \in S}{\arg\min}\, d_I(x,y) \neq \emptyset$. ii. $d_I(x,y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}$ $= \|x-y\|$. If $\lambda \in [0,1]$, $\forall x \in X, y_1, y_2 \in S$, $\|x - (\lambda y_1 + (1-\lambda) y_2)\| =$

$= \|\lambda x - \lambda y_1 + (1-\lambda)x - (1-\lambda)y_2\| \leq \|\lambda x - \lambda y_1\| + \|(1-\lambda)x - (1-\lambda)y_2\| =$
$= \lambda \|x - y_1\| + (1-\lambda)\|x - y_2\|$, while due to the convexity of S, $\lambda y_1 + (1-\lambda) y_2 \in S$. Hence, $\forall x \in X$, $d_I(x,\cdot) : S \to \mathbb{R}$ is convex, defined on a convex domain, implying that $\underset{y \in S}{\arg\min}\, d_I(x,y)$ is a singleton, hence r is well defined. iii. The optional exercise 3, the compactness of S, and the joint continuity of $d_I(\cdot,\cdot)$ imply that $\forall x \in X$, $\forall x_n \to x$, $r(x_n) = \underset{y \in S}{\arg\min}\, d_I(x_n, y) \to \underset{y \in S}{\arg\min}\, d_I(x,y) = r(x)$, hence r is continuous. iv. If $x \in S$ then $\inf_{y \in S} d_I(x,y) = \min_{y \in S} d_I(x,y) = d_I(x,x) = 0$ which implies that $r(x) = \underset{y \in S}{\arg\min}\, d_I(x,y) = x$ since d_I is a proper metric.

Hence r is a retraction. $\square$

For the following result, $X = O_{d_1}[0_{\mathbb{R}^n}, 1] = \{x \in \mathbb{R}^n : d(0,x) \leq 1\}$ and S^{n-1} is the unit sphere, i.e. $S^{n-1} = \{x \in \mathbb{R}^n : d_1(x,0) = 1\}$.

(e.g. when $n=1$, then $O_{d_1}[0,1] = [-1,1]$ and $S^0 = \{-1,1\}$).

Borsuk lemma. S^{n-1} is not a retract of X.

Very Vague Sketch of Proof. The lemma can be proven via the use of notions residing in the field of algebraic topology and more precisely via the theory of singular homology. In a very imprecise manner if Y is an appropriate topological space then the sequence of $\mathbb{Z}$-modules (vector-like spaces over the integers) $(H_n(Y))_{n \in \mathbb{N}}$ codify in an algebraic manner topological properties of Y. Something like the following holds when $Y = X$ or S (above) $\mathcal{H}_n(Y) = \mathbb{Z}^{p_n}$ where

$$p_n = \begin{cases} \text{number of connected components of } Y, & n=0 \\ \text{number of } n\text{-dimensional holes of } Y, & n>0. \end{cases}$$

$\mathcal{H}_n$ is called the singular homology group of order n of Y. Given this, at least intuitively we can see that

$$\mathcal{H}_n(O_{d_1}[0_{\mathbb{R}^n},1]) = \begin{cases} \mathbb{Z}, & n=0 \\ \{0\}, & n>0 \end{cases}, \quad \mathcal{H}_n(S^{n-1}) = \begin{cases} \mathbb{Z}^2, & n=0, n=1 \\ \mathbb{Z}, & n=0, n>1. \\ \mathbb{Z}, & n=n \end{cases}$$

Thereby we have that $\mathcal{H}_n(O_{d_1}[0_{\mathbb{R}^n},1]) \neq \mathcal{H}_n(S^{n-1})$, $\forall n \geq 1$ (actually this "inequality" holds in the more abstract fashion of non-isomorpism).

It is also possible to prove that if S^{n-1} is a retract of $\bigcirc_{d_I}[0_{n\times 1},1]$, than then there exists a "linear" injection from $H_n(S^{n-1}) = \mathbb{Z}$ to $H_n(\bigcirc_{d_I}[0_{n\times 1},1]) = \{0\}$.

This is impossible since the only possible function $h: \mathbb{Z} \to \{0\}$ is $h(x) = 0$, $\forall x \in \mathbb{Z}$ which is obviously not an injection. Hence S^{n-1} cannot be a retract of $\bigcirc_{d_I}[0_{n\times 1},1]$. $\square$

We are ready to state and prove the Brouwer FPT.

The notes are in a state of perpetual correction. They do not substitute the lectures. Please report any typos to stelios@aueb.gr or the course's e-class.]